

Noncommutative motives and their applications

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The classical theory of pure motives (Grothendieck)

- \mathcal{V}_k category of **smooth projective** varieties over a field k ;
morphisms of varieties
- (Pure) Motives over k : linearization and idempotent completion
(+ inverting the Lefschetz motive)
- **Correspondences**: $\text{Corr}_{\sim, F}(X, Y)$: F -linear combinations of algebraic cycles $Z \subset X \times Y$ of codimension = $\dim X$
- composition of correspondences:

$$\text{Corr}(X, Y) \times \text{Corr}(Y, Z) \rightarrow \text{Corr}(X, Z)$$

$$(\pi_{X,Z})_*(\pi_{X,Y}^*(\alpha) \bullet \pi_{Y,Z}^*(\beta))$$

intersection product in $X \times Y \times Z$

- **Equivalence relations** on cycles: rational (or “algebraic”), homological, numerical
 - $\alpha \sim_{rat} 0$ if $\exists \beta$ correspondence in $X \times \mathbb{P}^1$ with $\alpha = \beta(0) - \beta(\infty)$ (moving lemma; Chow groups; Chow motives)
 - $\alpha \sim_{hom} 0$: vanishing under a chosen Weil cohomology functor H^*
 - $\alpha \sim_{num} 0$: trivial intersection number with every other cycle

The category of motives has different properties depending on the choice of the equivalence relation on correspondences

Effective motives Category $\text{Mot}_{\sim, F}^{\text{eff}}(k)$:

- Objects: (X, p) smooth projective variety X and idempotent $p^2 = p$ in $\text{Corr}_{\sim, F}(X, X)$
- Morphisms:

$$\text{Hom}_{\text{Mot}_{\sim, F}^{\text{eff}}(k)}((X, p), (Y, q)) = q\text{Corr}_{\sim, F}(X, Y)p$$

- tensor structure $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$
- notation $h(X)$ or $M(X)$ for the motive (X, id)

Tate motives

- \mathbb{L} Lefschetz motive: $h(\mathbb{P}^1) = 1 \oplus \mathbb{L}$ with $1 = h(\text{Spec}(k))$.
- formal inverse $\mathbb{L}^{-1} = \text{Tate motive}$; notation $\mathbb{Q}(1)$

Motives Category $\text{Mot}_{\sim}(k)$

- Objects: $(X, p, m) := (X, p) \otimes \mathbb{L}^{-m} = (X, p) \otimes \mathbb{Q}(m)$
- Morphisms:

$$\text{Hom}_{\text{Mot}_{\sim}(k)}((X, p, m), (Y, q, n)) = q\text{Corr}_{\sim, F}^{m-n}(X, Y)p$$

shifts the codimension of cycles (Tate twist)

- Chow motives; homological motives; numerical motives

Jannsen's semi-simplicity result

Theorem (Jannsen 1991): TFAE

- $\text{Mot}_{\sim, F}(k)$ is a semi-simple *abelian* category
- $\text{Corr}_{\sim, F}^{\dim X}(X, X)$ is a finite-dimensional semi-simple F -algebra, for each X
- The equivalence relation is numerical: $\sim = \sim_{\text{num}}$

Weil cohomologies $H^* : \mathcal{V}_k^{op} \rightarrow \text{VecGr}_F$

- Künneth formula: $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$
- $\dim H^2(\mathbb{P}^1) = 1$; Tate twist: $V(r) = V \otimes H^2(\mathbb{P}^1)^{\otimes -r}$
- trace map (Poincaré duality) $tr : H^{2d}(X)(d) \rightarrow F$
- cycle map $\gamma_n : \mathcal{L}^n(X)_F \rightarrow H^{2n}(X)(n)$ (algebraic cycles to cohomology classes)

Examples: deRham, Betti, ℓ -adic étale

Grothendieck's idea of motives: universal cohomology theory for algebraic varieties lying behind all realizations via Weil cohomologies

Also recall: **Grothendieck's standard conjectures** of type C and D

- (Künneth) **C**: The Künneth components of the diagonal Δ_X are algebraic
 - (Hom=Num) **D** Homological and numerical equivalence coincide
- (Also **B**: Lefschetz involution algebraic; **I** Hodge involution pos def quadratic form on alg cycles with homological eq)

Motivic Galois groups

More structure than abelian category: **Tannakian** category $\text{Rep}_F(G)$
fin dim lin reps of an affine group scheme G

- F -linear, abelian, tensor category (*symmetric monoidal*)

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

- functorial isomorphisms:

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$$

$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X \quad \text{with} \quad c_{X,Y} \circ c_{Y,X} = 1_{X \otimes Y}$$

$$l_X : X \otimes 1 \xrightarrow{\cong} X, \quad r_X : 1 \otimes X \xrightarrow{\cong} X$$

- **Rigid**: duality $\vee : \mathcal{C} \rightarrow \mathcal{C}^{op}$ with $\epsilon : X \otimes X^\vee \rightarrow 1$ and $\eta : 1 \rightarrow X^\vee \otimes X$

$$X \simeq X \otimes 1 \xrightarrow{1_X \otimes \eta} X \otimes X^\vee \otimes X \xrightarrow{\epsilon \otimes 1_X} 1 \otimes X \simeq X$$

composition is identity

- categorical trace (Euler characteristic)

$$tr(f) = \epsilon \circ c_{X^v \otimes X} \circ (1_{X^v} \otimes f) \circ \eta; \dim X = tr(1_X)$$

- **Tannakian**: as above (and with $\text{End}(1) = F$) and *fiber functor*

$$\omega : \mathcal{C} \rightarrow \text{Vect}(K)$$

$K = \text{extension of } F; \omega \text{ exact faithful tensor functor; neutral Tannakian if } K = F$

- equivalence $\mathcal{C} \simeq \text{Rep}_F(G)$, affine group scheme

$$G = \text{Gal}(\mathcal{C}) = \underline{\text{Aut}}^{\otimes}(\omega)$$

- Deligne's characterization (char 0): Tannakian iff $tr(1_X)$ non-negative for all X

Tannakian categories and standard conjectures

In the case of $\text{Mot}_{\sim_{num}}(k)$, when Tannakian?

- problem: $\text{tr}(1_X) = \chi(X)$ Euler characteristic can be negative
- $\text{Mot}_{\sim_{num}}^\dagger(k)$ category $\text{Mot}_{\sim_{num}}(k)$ with modified commutativity constraint $c_{X,Y}$ by the Koszul sign rule (corrects for signs in the Euler characteristic)
- (Jannsen) if standard conjecture C (Künneth) holds then $\text{Mot}_{\sim_{num}}^\dagger(k)$ is Tannakian
- If conjecture D also holds then H^* fiber functor

Motives and Noncommutative motives

- Motives (pure): smooth projective algebraic varieties X
cohomology theories H_{dR} , H_{Betti} , H_{etale} , . . .
universal cohomology theory: motives \Rightarrow realizations
- NC Motives (pure): smooth proper dg-categories \mathcal{A}
homological invariants: K -theory, Hochschild and cyclic cohomology
universal homological invariant: NC motives

dg-categories

\mathcal{A} category whose morphism sets $\mathcal{A}(x, y)$ are complexes of k -modules ($k =$ base ring or field) with composition satisfying Leibniz rule

$$d(f \circ g) = df \circ g + (-1)^{\deg(f)} f \circ dg$$

dgcats = category of (small) dg-categories with dg-functors (preserving dg-structure)

From varieties to dg-categories

$$X \Rightarrow \mathcal{D}_{\text{perf}}^{\text{dg}}(X)$$

dg-category of perfect complexes

H^0 gives derived category $\mathcal{D}_{\text{perf}}(X)$ of perfect complexes of \mathcal{O}_X -modules

(loc quasi-isom to finite complexes of loc free sheaves of fin rank)

saturated dg-categories (Kontsevich)

- smooth dgcats: perfect as a bimodule over itself
- proper dgcats: if the complexes $\mathcal{A}(x, y)$ are perfect
- saturated = smooth + proper

smooth projective variety $X \Rightarrow$ smooth proper dgcats $\mathcal{D}_{\text{perf}}^{\text{dg}}(X)$
(but also smooth proper dgcats not from smooth proj varieties)

derived Morita equivalences

- \mathcal{A}^{op} same objects and morphisms $\mathcal{A}^{op}(x, y) = \mathcal{A}(y, x)$; right dg \mathcal{A} -module: dg-functor $\mathcal{A}^{op} \rightarrow \mathcal{C}_{dg}(k)$ (dg-cat of complexes of k -modules); $\mathcal{C}(\mathcal{A})$ cat of \mathcal{A} -modules; $\mathcal{D}(\mathcal{A})$ (derived cat of \mathcal{A}) localization of $\mathcal{C}(\mathcal{A})$ w/ resp to quasi-isom
- functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is derived Morita equivalence iff induced functor $\mathcal{D}(\mathcal{B}) \rightarrow \mathcal{D}(\mathcal{A})$ (restriction of scalars) is an equivalence of triangulated categories
- cohomological invariants (K -theory, Hochschild and cyclic cohomologies) are derived Morita invariant: send derived Morita equivalences to isomorphisms

symmetric monoidal category \mathbf{Hmo}

- homotopy category: dg-categories up to derived Morita equivalences
- \otimes extends from k -algebras to dg-categories
- can be derived with respect to derived Morita equivalences (gives symmetric monoidal structure on \mathbf{Hmo})
- saturated dg-categories = dualizable objects in \mathbf{Hmo} (Cisinski–Tabuada)

Further refinement: Hmo_0

- all cohomological invariants listed are “additive invariants”:

$$E : \text{dgcats} \rightarrow A, \quad E(\mathcal{A}) \oplus E(\mathcal{B}) = E(|M|)$$

where A additive category and $|M|$ dg-category

$\text{Obj}(|M|) = \text{Obj}(\mathcal{A}) \cup \text{Obj}(\mathcal{B})$ morphisms $\mathcal{A}(x, y)$, $\mathcal{B}(x, y)$,
 $X(x, y)$ with X a \mathcal{A} - \mathcal{B} bimodule

- Hmo_0 : objects dg-categories, morphisms $K_0\text{rep}(\mathcal{A}, \mathcal{B})$ with $\text{rep}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\text{op}} \otimes^{\mathbb{L}} \mathcal{B})$ full triang subcat of \mathcal{A} - \mathcal{B} bimodules X with $X(a, -) \in \mathcal{D}_{\text{perf}}(\mathcal{B})$; composition = (derived) tensor product of bimodules
- (Tabuada) $\mathcal{U}_A : \text{dgcats} \rightarrow \text{Hmo}_0$, id on objects, sends dg-functor to class in Grothendieck group of associated bimodule (\mathcal{U}_A characterized by a universal property)
- all additive invariants factor through Hmo_0

noncommutative Chow motives (Kontsevich) $\text{NChow}_F(k)$

- $\text{Hm}_{0;F} =$ the F -linearization of additive category Hm_{00}
- $\text{Hm}_{0;F}^{\natural} =$ idempotent completion of $\text{Hm}_{0;F}$
- $\text{NChow}_F(k) =$ idempotent complete full subcategory gen by saturated dg-categories

$\text{NChow}_F(k)$:

- Objects: (\mathcal{A}, e) smooth proper dg-categories (and idempotents)
- Morphisms $K_0(\mathcal{A}^{op} \otimes_k^{\mathbb{L}} \mathcal{B})_F$ (correspondences)
- Composition: induced by derived tensor product of bimodules

relation to commutative Chow motives (Tabuada):

$$\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}_{\mathbb{Q}}(k)$$

commutative motives embed as noncommutative motives after moding out by the Tate motives

orbit category $\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)}$

$(\mathcal{C}, \otimes, \mathbf{1})$ additive, F -linear, rigid symmetric monoidal;

$\mathcal{O} \in \mathrm{Obj}(\mathcal{C})$ \otimes -invertible object:

orbit category $\mathcal{C}/_{-\otimes\mathcal{O}}$ same objects and morphisms

$$\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(X, Y) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes j})$$

Numerical noncommutative motives

M.M., G.Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity*, arXiv:1105.2950, American J. Math. (to appear)

(\mathcal{A}, e) and (\mathcal{B}, e') objects in $\text{NChow}_F(k)$ and correspondences

$$\underline{X} = e \circ \left[\sum_i a_i X_i \right] \circ e', \quad \underline{Y} = e' \circ \left[\sum_j b_j Y_j \right] \circ e$$

X_i and Y_j bimodules

\Rightarrow **intersection number**:

$$\langle \underline{X}, \underline{Y} \rangle = \sum_{ij} [HH(\mathcal{A}; X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j)] \in K_0(k)_F$$

with $[HH(\mathcal{A}; X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j)]$ class in $K_0(k)_F$ of Hochschild homology complex of \mathcal{A} with coefficients in the \mathcal{A} - \mathcal{A} bimodule $X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j$

numerically trivial: \underline{X} if $\langle \underline{X}, \underline{Y} \rangle = 0$ for all \underline{Y}

- \otimes -ideal \mathcal{N} in the category $\text{NChow}_F(k)$
- \mathcal{N} largest \otimes -ideal strictly contained in $\text{NChow}_F(k)$

numerical motives: $\text{NNum}_F(k)$

$$\text{NNum}_F(k) = \text{NChow}_F(k) / \mathcal{N}$$

Thm: **abelian semisimple** (M.M., G.Tabuada, arXiv:1105.2950)

- $\text{NNum}_F(k)$ is abelian semisimple

analog of Jannsen's result for commutative numerical pure motives

What about Tannakian structures and motivic Galois groups?

For commutative motives this involves standard conjectures (C = Künneth and D = homological and numerical equivalence)

Questions:

- is $\text{NNum}_F(k)$ (neutral) super-Tannakian?
- is there a good analog of the standard conjecture C (Künneth)?
- does this make the category Tannakian?
- is there a good analog of standard conjecture D (numerical = homological)?
- does this neutralize the Tannakian category?
- relation between motivic Galois groups for commutative and noncommutative motives?

Tannakian categories $(\mathcal{C}, \otimes, \mathbf{1})$

F -linear, abelian, rigid symmetric monoidal with $\text{End}(\mathbf{1}) = F$

- **Tannakian**: \exists K -valued *fiber functor*, K field ext of F : exact faithful \otimes -functor $\omega : \mathcal{C} \rightarrow \text{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathcal{C} \simeq \text{Rep}_F(\text{Gal}(\mathcal{C}))$ affine group scheme (Galois group) $\text{Gal}(\mathcal{C}) = \underline{\text{Aut}}^{\otimes}(\omega)$

- **intrinsic characterization** (Deligne): F char zero, \mathcal{C} Tannakian iff $\text{Tr}(id_X)$ non-negative integer for each object X

super-Tannakian categories $(\mathcal{C}, \otimes, \mathbf{1})$

F -linear, abelian, rigid symmetric monoidal with $\text{End}(\mathbf{1}) = F$
 $s\text{Vect}(K)$ super-vector spaces $\mathbb{Z}/2\mathbb{Z}$ -graded

• **super-Tannakian**: \exists K -valued *super fiber functor*, K field ext of F :
exact faithful \otimes -functor $\omega : \mathcal{C} \rightarrow s\text{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathcal{C} \simeq \text{Rep}_F(s\text{Gal}(\mathcal{C}), \epsilon)$ super-reps of affine
super-group-scheme (super-Galois group)

$s\text{Gal}(\mathcal{C}) = \underline{\text{Aut}}^{\otimes}(\omega)$ $\epsilon =$ parity automorphism

• **intrinsic characterization** (Deligne) F char zero, \mathcal{C} super-Tannakian
iff Schur finite (if F alg closed then neutral super-Tannakian iff Schur
finite)

• **Schur finite**: symm grp S_n , idempotent $c_\lambda \in \mathbb{Q}[S_n]$ for partition λ of
 n (irreps of S_n), Schur functors $S_\lambda : \mathcal{C} \rightarrow \mathcal{C}$, $S_\lambda(X) = c_\lambda(X^{\otimes n})$
 \mathcal{C} = Schur finite iff all objects X annihilated by some Schur functor
 $S_\lambda(X) = 0$

Main results

M.M., G.Tabuada, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups*, arXiv:1110.2438

assume either: (i) $K_0(k) = \mathbb{Z}$, F is k -algebra; (ii) k and F both field extensions of a field K

- **Thm 1:** $\text{NNum}_F(k)$ is super-Tannakian; if F alg closed also neutral
- **Thm 2:** standard conjecture $C_{NC}(\mathcal{A})$: the Künneth projectors

$$\pi_{\mathcal{A}}^{\pm} : \overline{HP}_*(\mathcal{A}) \twoheadrightarrow \overline{HP}_*^{\pm}(\mathcal{A}) \hookrightarrow \overline{HP}_*(\mathcal{A})$$

are algebraic: $\pi_{\mathcal{A}}^{\pm} = \overline{HP}_*(\underline{\pi}_{\mathcal{A}}^{\pm})$ with $\underline{\pi}_{\mathcal{A}}^{\pm}$ correspondences. If k field ext of F char 0, sign conjecture implies

$$C^+(Z) \Rightarrow C_{NC}(\mathcal{D}_{perf}^{dg}(Z))$$

i.e. on commutative motives more likely to hold than sign conjecture

- **Thm 3:** k and F char 0, one extension of other: if C_{NC} holds then change of symmetry isomorphism in tensor structure gives category $\text{NNum}_F^\dagger(k)$ Tannakian

- **Thm 4:** standard conjecture $D_{NC}(\mathcal{A})$:

$$K_0(\mathcal{A})_F / \sim_{\text{hom}} = K_0(\mathcal{A})_F / \sim_{\text{num}}$$

homological defined by periodic cyclic homology: kernel of

$$K_0(\mathcal{A})_F = \text{Hom}_{\text{NChow}_F(k)}(k, \mathcal{A}) \xrightarrow{\overline{HP}_*} \text{Hom}_{\text{Vect}(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathcal{A}))$$

when k field ext of F char 0: $D(Z) \Rightarrow D_{NC}(\mathcal{D}_{\text{perf}}^{\text{dg}}(Z))$

i.e. for commutative motives more likely to hold than D conjecture

- **Thm 5:** F ext of k char 0: if C_{NC} and D_{NC} hold then $\text{NNum}_F^\dagger(k)$ is a neutral Tannakian category with periodic cyclic homology as fiber functor
- **Thm 6:** k char 0: if C , D and C_{NC} , D_{NC} hold then

$$\text{sGal}(\text{NNum}_k(k) \twoheadrightarrow \text{Ker}(t : \text{sGal}(\text{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

$$\text{Gal}(\text{NNum}_k^\dagger(k) \twoheadrightarrow \text{Ker}(t : \text{Gal}(\text{Num}_k^\dagger(k)) \twoheadrightarrow \mathbb{G}_m)$$

where t induced by inclusion of Tate motives in the category of (commutative) numerical motives

(using periodic cyclic homology and de Rham cohomology)

What is kernel? Ker = “truly noncommutative motives”

$$\mathrm{Gal}(\mathrm{NNum}_k^\dagger(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Num}_k^\dagger(k) \rightarrow \mathbb{G}_m)$$

$$\mathrm{sGal}(\mathrm{NNum}_k(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{sGal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look like? examples? general properties?

Are there truly noncommutative motives? Still an open question!

... but the theory of NC motives can be used as a new tool to study ordinary motives

Using NC motives to study commutative motives

Example: full exceptional collections and motivic decompositions

Examples of motivic decompositions:

- Projective spaces: $h(\mathbb{P}^n) = 1 \oplus \mathbb{L} \oplus \dots \oplus \mathbb{L}^n$
- Quadrics (k alg closed char 0):

$$h(Q_d)_{\mathbb{Q}} \simeq \begin{cases} 1 \oplus \mathbb{L} \oplus \dots \oplus \mathbb{L}^{\otimes n} & d \text{ odd} \\ 1 \oplus \mathbb{L} \oplus \dots \oplus \mathbb{L}^{\otimes n} \oplus \mathbb{L}^{\otimes(d/2)} & d \text{ even.} \end{cases}$$

- Fano 3-folds:

$$h(X)_{\mathbb{Q}} \simeq 1 \oplus h^1(X) \oplus \mathbb{L}^{\oplus b} \oplus (h^1(J) \otimes \mathbb{L}) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b} \oplus h^5(X) \oplus \mathbb{L}^{\otimes 3},$$

$h^1(X)$ and $h^5(X)$ Picard and Albanese motives, $b = b_2(X) = b_4(X)$
 J abelian variety

Full exceptional collections in the derived category $\mathcal{D}^b(X)$

A collection of objects $\{E_1, \dots, E_n\}$ in a F -linear triangulated category \mathcal{C} is *exceptional* if $\mathrm{RHom}(E_i, E_i) = F$ for all i and $\mathrm{RHom}(E_i, E_j) = 0$ for all $i > j$; it is *full* if \mathcal{C} is minimal triangulated subcategory containing it.

Examples of full exceptional collections:

- Projective spaces (Beilinson): $(\mathcal{O}(-n), \dots, \mathcal{O}(0))$
- Quadrics (Kapranov):

$$(\Sigma(-d), \mathcal{O}(-d+1), \dots, \mathcal{O}(-1), \mathcal{O}) \quad \text{if } d \text{ is odd}$$

$$(\Sigma_+(-d), \Sigma_-(-d), \mathcal{O}(-d+1), \dots, \mathcal{O}(-1), \mathcal{O}) \quad \text{if } d \text{ is even,}$$

Σ_{\pm} (and Σ) spinor bundles

- Toric varieties (Kawamata)
 - Homogeneous space (Kuznetsov-Polishchuk)
- Conjecture (KP): k alg cl char 0, parabolic subgroup $P \subset G$ of semisimple alg group then $\mathcal{D}^b(G/P)$ has full exceptional collection
- Fano 3-folds with vanishing odd cohomology (Ciolli)
 - Moduli spaces of rational curves $\overline{\mathcal{M}}_{0,n}$ (Manin–Smirnov)

Note: all these cases also have motivic decompositions

Deeper reason: exceptional collections and motivic decompositions are related through the relation between commutative and NC motives

Thm 7: Full exceptional collections and motivic decompositions

if $\mathcal{D}^b(X)$ has a full exceptional collection, then $h(X)_{\mathbb{Q}}$ has a motivic decomposition

$$h(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\ell_1} \oplus \dots \oplus \mathbb{L}^{\ell_m}$$

for some $\ell_1, \dots, \ell_m \geq 0$

(Note: works also for Deligne–Mumford stacks)

- $\mathcal{D}_{dg}^b(X)$ unique dg enhancement: $\langle E_j \rangle_{dg} \simeq \mathcal{D}_{dg}^b(k)$
- Look at corresponding elements in $\text{NChow}_{\mathbb{Q}}(k)$ under universal localizing invariant $\mathcal{U} : \text{dgcats}(k) \rightarrow \text{NChow}_{\mathbb{Q}}(k)$

$$\bigoplus_{j=1}^m \mathcal{U}(\mathcal{D}_{dg}^b(k)) \xrightarrow{\cong} \mathcal{U}(\mathcal{D}_{dg}^b(X))$$

from inclusions of dg categories $\langle E_j \rangle_{dg} \hookrightarrow \mathcal{D}_{dg}^b(X)$

using (Tabuada “Higher K-theory via universal invariants”): given split short exact sequence of pre-triangulated dg categories

$$0 \longrightarrow \mathcal{B} \xrightarrow{\nu_{\mathcal{B}}} \mathcal{A} \xrightarrow{\nu_{\mathcal{C}}} \mathcal{C} \longrightarrow 0$$

$\xleftarrow{\quad}$ above $\mathcal{B} \rightarrow \mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{C}$

mapped by universal localizing invariant $\mathcal{U}(-)$ to a distinguished split triangle so $\mathcal{U}(\mathcal{B}) \oplus \mathcal{U}(\mathcal{C}) \xrightarrow{\sim} \mathcal{U}(\mathcal{A})$

Applied to

$$\mathcal{A} := \langle E_i, \dots, E_m \rangle_{dg}, \quad \mathcal{B} := \langle E_i \rangle_{dg}, \quad \mathcal{C} := \langle E_{i+1}, \dots, E_m \rangle_{dg}$$

gives

$$\mathcal{U}(\mathcal{D}_{dg}^b(k)) \oplus \mathcal{U}(\langle E_{i+1}, \dots, E_m \rangle_{dg}) \xrightarrow{\sim} \mathcal{U}(\langle E_i, \dots, E_m \rangle_{dg})$$

recursively get result using $\mathcal{D}_{dg}^b(X) = \langle E_1, \dots, E_m \rangle_{dg}$

A consequence: **Hodge–Tate cohomology**

Thm 8: If a smooth complex projective variety V has a full exceptional collection then it is Hodge–Tate (Hodge numbers $h^{p,q}(V) = 0$ for $p \neq q$)

Reason: motivic decomposition

Dubrovin conjecture: V smooth projective complex

(i) Quantum cohomology of V is (generically) semi-simple if and only if V is Hodge-Tate and $\mathcal{D}^b(V)$ has a full exceptional collection.

(ii) Stokes matrix of structure connection of quantum cohomology = Gram matrix of exceptional collection

$$\chi : K_0(V) \times K_0(V) \rightarrow \mathbb{Z}, \quad \sum_{n \in \mathbb{Z}} (-1)^n \dim \text{Ext}^n(\mathcal{F}_1, \mathcal{F}_2)$$

First observation: Hodge-Tate hypothesis not necessary

NC-motivic approach to the Dubrovin conjecture? currently work in progress...

Jacobians of noncommutative motives

- Jacobians of curves $J(C)$: geometric model for cohomology $H^1(C)$, one of the origins of the theory of motives (Weil)
- Smooth projective X : Picard and Albanese varieties $Pic^0(X)$ and $Alb(X)$ geometric models for $H^1(X)$ and $H^{2d-1}(X)$
- Griffiths intermediate Jacobians ($F =$ Hodge filtration)

$$J_i(X) := \frac{H^{2i+1}(X, \mathbb{C})}{F^{i+1}H^{2i+1}(X, \mathbb{C}) + H^{2i+1}(X, \mathbb{Z})}$$

not algebraic but $J_i^a(X) \subseteq J_i(X)$ algebraic: image of Abel-Jacobi

$$AJ_i : CH^{i+1}(X)_{\mathbb{Z}}^0 \rightarrow J_i(X)$$

with $CH^{i+1}(X)_{\mathbb{Z}}^0$ group of alg.-trivial cycles codim $i + 1$
(see recent work of Charles Vial)

- Know how to go from commutative to noncommutative motives via

$$\mathrm{Chow}(k)_{\mathbb{Q}} / - \otimes_{\mathbb{Q}}(1) \hookrightarrow \mathrm{NChow}(k)_{\mathbb{Q}}$$

- **Question:** can one go the other way? Assign functorially a “commutative part” to a noncommutative motive?
- **Idea:** a theory of Jacobians for NC motives

$$\mathrm{NChow}(k)_{\mathbb{Q}} \rightarrow \mathrm{Ab}(k)_{\mathbb{Q}}, \quad N \mapsto \mathbf{J}(N)$$

\mathbb{Q} -linear additive Jacobian functor to category $\mathrm{Ab}(k)_{\mathbb{Q}}$ of abelian varieties up to isogeny

- Periodic cyclic homology

$$HP^\pm : \text{NChow}(k)_\mathbb{Q} \rightarrow \text{sVect}(k)$$

- Piece of HP generated by curves

$$HP_{\text{curves}}^-(N) := \sum_{C, \Gamma} \text{Im}(HP^-(\text{perf}(C)) \xrightarrow{HP^-(\Gamma)} HP^-(N))$$

$C =$ smooth projective curve; $\Gamma : \text{perf}(C) \rightarrow N$ a morphism (correspondence) in $\text{NChow}(k)_\mathbb{Q}$

Results (MM, G. Tabuada, arXiv:1212.1118)

Thm 9:

- k char zero, have \mathbb{Q} -additive linear functor

$$\mathrm{NChow}(k)_{\mathbb{Q}} \rightarrow \mathrm{Ab}(k)_{\mathbb{Q}}, \quad N \mapsto \mathbf{J}(N)$$

- $\forall N \in \mathrm{NChow}(k)_{\mathbb{Q}}$ there is C_N smooth proj curve and $\Gamma_N : \mathrm{perf}(C_N) \rightarrow N$ with

$$H_{dR}^1(\mathbf{J}(N)) = \mathrm{Im} HP^-(\Gamma_N)$$

so $H_{dR}^1(\mathbf{J}(N)) \subseteq HP_{\mathrm{curves}}^-(N)$

- if conjecture D_{NC} holds for $\mathrm{perf}(C) \otimes N$, for smooth proj curves C ,

$$H_{dR}^1(\mathbf{J}(N)) = HP_{\mathrm{curves}}^-(N)$$

- for smooth projective X let

$$NH_{dR}^{2i+1}(X) := \sum_{C, \gamma_i} \text{Im}(H_{dR}^1(C) \xrightarrow{H_{dR}(\gamma_i)} H_{dR}^{2i+1}(X))$$

with $\gamma_i : M(C) \rightarrow M(X)(i)$ morphism in $\text{Chow}(k)_{\mathbb{Q}}$

- Intersection bilinear pairing restricted to these ($0 \leq i \leq d-1$)

$$\langle -, - \rangle : NH_{dR}^{2d-2i-1}(X) \times NH_{dR}^{2i+1}(X) \rightarrow k$$

- **Thm 10:** if $k = \bar{k} \subseteq \mathbb{C}$ and X smooth projective and if pairings above are *nondegenerate* then

$$\mathbf{J}(\text{perf}(X)) = \prod_{i=0}^{d-1} J_i^a(X)$$

and $H_{dR}^1(\mathbf{J}(\text{perf}(X))) \otimes_k \mathbb{C} = \bigoplus_{i=0}^{d-1} NH_{dR}^{2i+1}(X) \otimes_k \mathbb{C}$

Sketch of argument on NC Jacobians: construction of $\mathbf{J}(N)$

- Categories of NC motives: $\mathrm{NChow}(k)_{\mathbb{Q}}$, $\mathrm{NHomo}(k)_{\mathbb{Q}}$, $\mathrm{NNum}(k)_{\mathbb{Q}}$
- $\mathrm{NNum}(k)_{\mathbb{Q}}$ is abelian semi-simple: $N = S_1 \oplus \cdots \oplus S_n$ unique finite decomposition into simple objects
- classical motives: $\mathrm{Homo}(k)_{\mathbb{Q}} \supset \{\pi^1 M(C)\}^{\natural} = \mathrm{Ab}(k)_{\mathbb{Q}}$ and same in $\mathrm{Num}(k)_{\mathbb{Q}} \supset \{\pi^1 M(C)\}^{\natural} = \mathrm{Ab}(k)_{\mathbb{Q}}$
- functor mapping $\mathrm{Ab}(k)_{\mathbb{Q}}$ to $\mathrm{NNum}(k)_{\mathbb{Q}}$ with image $\overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$

$$\mathrm{Ab}(k)_{\mathbb{Q}} \rightarrow \mathrm{Num}(k)_{\mathbb{Q}} \rightarrow \mathrm{Num}(k)_{\mathbb{Q}} / \! - \otimes_{\mathbb{Q}} \mathbb{Q}(1) \rightarrow \mathrm{NNum}(k)_{\mathbb{Q}}$$

- $\mathrm{Ab}(k)_{\mathbb{Q}} \simeq \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$ equivalence of categories
- $\mathcal{S} =$ simple objects of $\mathrm{NNum}(k)_{\mathbb{Q}}$ belonging to $\overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$
- truncation functor $\mathrm{NNum}(k)_{\mathbb{Q}} \rightarrow \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$, with $N \mapsto \tau(N)$ only simple objects in \mathcal{S} of decomposition of N

properties of functor $N \mapsto \mathbf{J}(N)$

- because $\text{Ab}(k)_{\mathbb{Q}} \simeq \overline{\text{Ab}}(k)_{\mathbb{Q}}$ every object in $\overline{\text{Ab}}(k)_{\mathbb{Q}}$ is a direct factor of some $\pi^1 \text{perf}(C)$
- so get C_N for any $N \in \text{NNum}(k)_{\mathbb{Q}}$ through $\tau(N) \in \overline{\text{Ab}}(k)_{\mathbb{Q}}$
- and correspondence Γ_N giving $\tau(N)$ as direct factor of $\pi^1 \text{perf}(C_N)$ and this as direct factor of $\text{perf}(C_N)$
- $H_{dR}^1(C_N) = HP^-(\text{perf}(C_N)) = HP^-(\pi^1 \text{perf}(C_N)) \xrightarrow{HP^-(\Gamma_N)} HP^-(N)$
- $HP^-(\pi^1 \text{perf}(C_N)) \xrightarrow{HP^-(\bar{\Gamma}_N)} HP^-(\tau(N))$ surjective and $HP^-(\tau(N)) \rightarrow HP^-(N)$ from $\tau(N) \hookrightarrow N$ injective \Rightarrow $HP^-(\tau(N)) = \text{Im}(HP^-(\bar{\Gamma}_N))$ and $H_{dR}^1(\mathbf{J}(N)) = \text{Im}(HP^-(\bar{\Gamma}_N))$
- If $D_{NC}(\text{perf}(C) \otimes N)$ holds then as \mathbb{Q} -vector spaces

$$\text{Hom}_{\text{NHomo}(k)_{\mathbb{Q}}}(\text{perf}(C), N) = \text{Hom}_{\text{NNum}(k)_{\mathbb{Q}}}(\text{perf}(C), N)$$

applying HP^- : morphism $HP^-(\Gamma)$ factors through $HP^-(\tau(N))$ for all C, Γ , so obtain $HP^-(\tau(N)) = HP_{\text{curves}}^-(N)$

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More details on the category of NC motives:

Thm 1: **Schur finiteness** $\overline{HH} : \text{NChow}_F(k) \rightarrow \mathcal{D}_c(F)$

F -linear symmetric monoidal functor (Hochschild homology)

$$(\text{NChow}_F(k)/\text{Ker}(\overline{HH}))^{\natural} \rightarrow \mathcal{D}_c(F)$$

faithful F -linear symmetric monoidal

$\mathcal{D}_c(\mathcal{A}) =$ full triang subcat of compact objects in $\mathcal{D}(\mathcal{A}) \Rightarrow \mathcal{D}_c(F)$

identified with fin-dim \mathbb{Z} -graded F -vector spaces: Schur finite

general fact: $L : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ F -linear symmetric monoidal functor:

$X \in \mathcal{C}_1$ Schur finite $\Rightarrow L(X) \in \mathcal{C}_2$ Schur finite; L faithful then also

converse: $L(X) \in \mathcal{C}_2$ Schur finite $\Rightarrow X \in \mathcal{C}_1$ Schur finite

conclusion: $(\text{NChow}_F(k)/\text{Ker}(\overline{HH}))^{\natural}$ is Schur finite

also $\text{Ker}(\overline{HH}) \subset \mathcal{N}$ with F -linear symmetric monoidal functor

$(\text{NChow}_F(k)/\text{Ker}(\overline{HH}))^{\natural} \rightarrow (\text{NChow}_F(k)/\mathcal{N})^{\natural} = \text{NNum}_F(k)$

$\Rightarrow \text{NNum}_F(k)$ Schur finite \Rightarrow **super-Tannakian**

Thm 2: **periodic cyclic homology**

mixed complex (M, b, B) with $b^2 = B^2 = Bb + bB = 0$,
 $\deg(b) = 1 = -\deg(B)$: *periodized*

$$\cdots \prod_{n \text{ even}} M_n \xrightarrow{b+B} \prod_{n \text{ odd}} M_n \xrightarrow{b+B} \prod_{n \text{ even}} M_n \cdots$$

periodic cyclic homology (the derived cat of $\mathbb{Z}/2\mathbb{Z}$ -graded complexes

$$HP : \text{dgcats} \rightarrow \mathcal{D}_{\mathbb{Z}/2\mathbb{Z}}(k)$$

induces F -linear symmetric monoidal functor

$$\overline{HP}_* : \text{NChow}_F(k) \rightarrow \text{sVect}(F)$$

or to $\text{sVect}(k)$ if k field ext of F

Note the issue here:

- mixed complex functor symmetric monoidal but 2-periodization not (infinite product don't commute with \otimes)
 - *lax symmetric monoidal* with $\mathcal{D}_{\mathbb{Z}/2\mathbb{Z}}(k) \simeq \mathbf{SVect}(k)$ (not fin dim)
 - $HP : \text{dgcats} \rightarrow \mathbf{SVect}(k)$ *additive invariant*: through $\text{Hm}_{\mathbb{O}_0}(k)$
 - $\text{NChow}_F(k) = (\text{Hm}_{\mathbb{O}_0}(k)^{\text{sp}})_F^{\sharp}$ (sp = gen by smooth proper dgcats)
 - periodic cyclic hom *finite dimensional* for smooth proper dgcats + a result of Emmanouil
- \Rightarrow lax symmetric monoidal $\overline{HP}_* : \text{Hm}_{\mathbb{O}_0}(k)^{\text{sp}} \rightarrow \mathbf{sVect}(k)$ is symmetric monoidal

standard conjecture C_{NC} (Künneth type)

- $C_{NC}(\mathcal{A})$: Künneth projections

$$\pi_{\mathcal{A}}^{\pm} : \overline{HP}_*(\mathcal{A}) \twoheadrightarrow \overline{HP}_*^{\pm}(\mathcal{A}) \hookrightarrow \overline{HP}_*(\mathcal{A})$$

are algebraic: $\pi_{\mathcal{A}}^{\pm} = \overline{HP}_*(\underline{\pi}_{\mathcal{A}}^{\pm})$ image of correspondences

- then from Keller + Hochschild-Konstant-Rosenberg have $\overline{HP}_*(\mathcal{D}_{perf}^{dg}(Z)) = HP_*(\mathcal{D}_{perf}^{dg}(Z)) = HP_*(Z) = \bigoplus_{n \text{ even/odd}} H_{dR}^n(Z)$

- hence $C^+(Z) \Rightarrow C_{NC}(\mathcal{D}_{perf}^{dg}(Z))$ with $\pi_{\mathcal{D}_{perf}^{dg}(Z)}^{\pm}$ image of $\underline{\pi}_Z^{\pm}$ under $\text{Chow}(k) \rightarrow \text{Chow}(k)/_{-\otimes \mathbb{Q}(1)} \hookrightarrow \text{NChow}(k)$

classical: (using deRham as Weil cohomology) $C(Z)$ for Z correspondence, the Künneth projections $\pi_Z^n : H_{dR}^*(Z) \twoheadrightarrow H_{dR}^n(Z)$ are algebraic, $\pi_Z^n = H_{dR}^*(\underline{\pi}_Z^n)$, with $\underline{\pi}_Z^n$ correspondences

sign conjecture: $C^+(Z)$: Künneth projectors $\pi_Z^+ = \sum_{n=0}^{\dim Z} \pi_Z^{2n}$ are algebraic, $\pi_Z^+ = H_{dR}^*(\underline{\pi}_Z^+)$ (hence π_Z^- also)

Thm 3: Tannakian category first steps

- have F -linear symmetric monoidal and also full and essentially surjective functor: $\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*) \rightarrow \text{NChow}_F(k)/\mathcal{N}$
- assuming $C_{NC}(\mathcal{A})$: have $\pi_{(\mathcal{A}, e)}^\pm = e \circ \pi_{\mathcal{A}}^\pm \circ e$; if \underline{X} trivial in $\text{NChow}_F(k)/\mathcal{N}$ intersection numbers $\langle \underline{X}^n, \pi_{(\mathcal{A}, e)}^\pm \rangle$ vanishes (\mathcal{N} is \otimes -ideal)
- intersection number is categorical trace of $\underline{X}^n \circ \pi_{(\mathcal{A}, e)}^\pm$ (M.M., G.Tabuada, 1105.2950)

$$\Rightarrow \text{Tr}(\overline{HP}_*(\underline{X}^n \circ \pi_{(\mathcal{A}, e)}^\pm)) = \text{Tr}(\overline{HP}_*^\pm(\underline{X})^n) = 0$$

trace all n -compositions vanish \Rightarrow nilpotent $\overline{HP}_*^\pm(\underline{X})$

- conclude: nilpotent ideal as kernel of

$$\text{End}_{\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*)}(\mathcal{A}, e) \twoheadrightarrow \text{End}_{\text{NChow}_F(k)/\mathcal{N}}(\mathcal{A}, e)$$

- then functor $(\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*))^{\text{d}} \rightarrow \text{NNum}_F(k)$ full conservative essentially surjective: (quotient by \mathcal{N} full and ess surj; idempotents can be lifted along surj F -linear homom with nilpotent

Tannakian category: modification of tensor structure

- $H : \mathcal{C} \rightarrow \mathbf{sVect}(K)$ symmetric monoidal F -linear (K ext of F) faithful, Künneth projectors $\pi_N^\pm = H(\underline{\pi}_N^\pm)$ for $\underline{\pi}_N^\pm \in \text{End}_{\mathcal{C}}(N)$ for all $N \in \mathcal{C}$ then modify symmetry isomorphism

$$c_{N_1, N_2}^\dagger = c_{N_1, N_2} \circ (e_{N_1} \otimes e_{N_2}) \quad \text{with } e_N = 2\underline{\pi}_N^+ - id_N$$

- get F -linear symmetric monoidal functor $\mathcal{C}^\dagger \xrightarrow{H} \mathbf{sVect}(K) \rightarrow \mathbf{Vect}(K)$
- if $P : \mathcal{C} \rightarrow \mathcal{D}$, F -linear symmetric monoidal (essentially) surjective, then $P : \mathcal{C}^\dagger \rightarrow \mathcal{D}^\dagger$ (use image of e_N to modify \mathcal{D} compatibly)
- apply to functors $\overline{HP}_* : (\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*))^\natural \rightarrow \mathbf{sVect}(K)$ and $(\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*))^\natural \rightarrow \text{NNum}_F(k)$

\Rightarrow obtain $\text{NNum}_F^\dagger(k)$ satisfying Deligne's intrinsic characterization for Tannakian: with \tilde{N} lift to $(\text{NChow}_F(k)/\text{Ker}(\overline{HP}_*))^{\natural, \dagger}$ have

$$\text{rk}(N) = \text{rk}(\overline{HP}_*(\tilde{N})) = \dim(\overline{HP}_*^+(\tilde{N})) + \dim(\overline{HP}_*^-(\tilde{N})) \geq 0$$

Thm 4: Noncommutative homological motives

$$\overline{HP}_* : \text{NChow}_F(k) \rightarrow \text{sVect}(K)$$

$$K_0(\mathcal{A})_F = \text{Hom}_{\text{NChow}_F(k)}(k, \mathcal{A}) \xrightarrow{\overline{HP}_*} \text{Hom}_{\text{sVect}(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathcal{A}))$$

kernel gives homological equivalence $K_0(\mathcal{A})_F \text{ mod } \sim_{\text{hom}}$

- $D_{\text{NC}}(\mathcal{A})$ standard conjecture:

$$K_0(\mathcal{A})_F / \sim_{\text{hom}} = K_0(\mathcal{A})_F / \sim_{\text{num}}$$

- on $\text{Chow}_F(k) / - \otimes_{\mathbb{Q}}(1)$ induces homological equivalence with sH_{dR} (de Rham even/odd) $\Rightarrow \mathcal{L}_{\text{hom}}^*(Z)_F \twoheadrightarrow K_0(\mathcal{D}_{\text{perf}}^{\text{dg}}(Z))_F / \sim_{\text{hom}}$

- **classical** cycles $\mathcal{L}_{\text{hom}}^*(Z)_F \simeq \mathcal{L}_{\text{num}}^*(Z)_F$; for numerical $\mathcal{L}_{\text{num}}^*(Z)_F \xrightarrow{\sim} K_0(\mathcal{D}_{\text{perf}}^{\text{dg}}(Z))_F / \sim_{\text{num}}$; then get

$$D(Z) \Rightarrow D_{\text{NC}}(\mathcal{D}_{\text{perf}}^{\text{dg}}(Z))$$

Thm 5: assume C_{NC} and D_{NC} then

$$\overline{HP}_* : \text{NNum}_F^\dagger(k) \rightarrow \text{Vect}(F)$$

exact faithful \otimes -functor: **fiber functor** \Rightarrow **neutral Tannakian category**
 $\text{NNum}_F^\dagger(k)$

Thm 6: **Motivic Galois groups**

- Galois group of neutral Tannakian category $\text{Gal}(\text{NNum}_F^\dagger(k))$ want to compare with commutative case $\text{Gal}(\text{Num}_F^\dagger(k))$
- super-Galois group of super-Tannakian category $\text{sGal}(\text{NNum}_F(k))$ compare with commutative motives case $\text{sGal}(\text{Num}_F(k))$
- related question: what are **truly noncommutative** motives?

Tate triples (Deligne–Milne)

- For $A = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{G}_m$ or μ_2 , Tannakian cat \mathcal{C} with A -grading: A -grading on objects with $(X \otimes Y)^a = \bigoplus_{a=b+c} X^b \otimes Y^c$; homom $w : B \rightarrow \underline{\text{Aut}}^{\otimes}(id_{\mathcal{C}})$ (weight); central hom $B \rightarrow \underline{\text{Aut}}^{\otimes}(\omega)$
- **Tate triple** (\mathcal{C}, w, T) : \mathbb{Z} -graded Tannakian \mathcal{C} with weight w , invertible object T (Tate object) weight -2
- Tate triple \Rightarrow central homom $w : \mathbb{G}_m \rightarrow \text{Gal}(\mathcal{C})$ and homom $t : \text{Gal}(\mathcal{C}) \rightarrow \mathbb{G}_m$ with $t \circ w = -2$.
- $H = \text{Ker}(t : \text{Gal}(\mathcal{C}) \rightarrow \mathbb{G}_m)$ defines Tannakian category $\simeq \text{Rep}(H)$. It is the “quotient Tannakian category” (Milne) of inclusion of subcategory gen by Tate object into \mathcal{C}

Galois group and orbit category

- $\mathcal{T} = (\mathcal{C}, w, T)$ Tate triple, $\mathcal{S} \subset \mathcal{C}$ gen by T , pseudo-ab envelope $(\mathcal{C}/_{-\otimes T})^{\natural}$ of orbit cat $\mathcal{C}/_{-\otimes T}$ is neutral Tannakian with

$$\mathrm{Gal}((\mathcal{C}/_{-\otimes T})^{\natural}) \simeq \mathrm{Ker}(t : \mathrm{Gal}(\mathcal{C}) \rightarrow \mathbb{G}_m)$$

- Quotient Tannakian categories with resp to a fiber functor (Milne): $\omega_0 : \mathcal{S} \rightarrow \mathrm{Vect}(F)$ then \mathcal{C}/ω_0 pseudo-ab envelope of \mathcal{C}' with same objects as \mathcal{C} and morphisms $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \omega_0(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^H)$ with X^H largest subobject where H acts trivially
- fiber functor $\omega_0 : X \mapsto \mathrm{colim}_n \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{r=-n}^n \mathbf{1}(r), X) \in \mathrm{Vect}(F)$
 \Rightarrow get $\mathcal{C}' = \mathcal{C}/_{-\otimes T}$

super-Tannakian case: super Tate triples

- Need a super-Tannakian version of Tate triples
- super Tate triple: $\mathcal{S} \mathcal{T} = (\mathcal{C}, \omega, \underline{\pi}^\pm, \mathcal{T}^\dagger)$ with $\mathcal{C} =$ neutral super-Tannakian; $\omega : \mathcal{C} \rightarrow \mathbf{sVect}(F)$ super-fiber functor; idempotent endos: $\omega(\underline{\pi}_X^\pm) = \pi_X^\pm$ Künneth proj.; neutral Tate triple $\mathcal{T}^\dagger = (\mathcal{C}^\dagger, w, T)$ with \mathcal{C}^\dagger modified symmetry constraint from \mathcal{C} using $\underline{\pi}^\pm$
- assuming C and D : a super Tate triple for (comm) num motives

$$(\mathrm{Num}_k(k), \overline{sH}_{dR}^*, \underline{\pi}_X^\pm, (\mathrm{Num}_k^\dagger(k), w, \mathbb{Q}(1)))$$

super-Tannakian case: orbit category

- $\mathcal{S}\mathcal{T} = (\mathcal{C}, \omega, \underline{\pi}^\pm, \mathcal{T}^\dagger)$ super Tate triple; $\mathcal{S} \subset \mathcal{C}$ full neutral super-Tannakian subcat gen by T
- Assume: $\underline{\pi}_T^-(T) = 0$; for $K = \text{Ker}(t : \text{Gal}(\mathcal{C}^\dagger) \rightarrow \mathbb{G}_m)$ of Tate triple \mathcal{T}^\dagger , if $\epsilon : \mu_2 \rightarrow H$ induced $\mathbb{Z}/2\mathbb{Z}$ grading from $t \circ w = -2$; then (H, ϵ) super-affine group scheme is Ker of $\text{sGal}(\mathcal{C}) \rightarrow \text{sGal}(\mathcal{S})$ and $\text{Rep}_F(H, \epsilon) = \text{Rep}_F^\dagger(H)$.
- Conclusion: pseudoabelian envelope of $\mathcal{C}/_{-\otimes T}$ is neutral super-Tannakian and seq of exact \otimes -functors $\mathcal{S} \subset \mathcal{C} \rightarrow (\mathcal{C}/_{-\otimes T})^{\text{h}}$ gives
$$\text{sGal}((\mathcal{C}/_{-\otimes T})^{\text{h}}) \xrightarrow{\sim} \text{Ker}(t : \text{sGal}(\mathcal{C}) \rightarrow \mathbb{G}_m)$$
- have also $(\mathcal{C}^\dagger/__{-\otimes T})^{\text{h}} \simeq (\mathcal{C}/_{-\otimes T})^{\text{h}, \dagger} \simeq \text{Rep}_F^\dagger(H, \epsilon) \simeq \text{Rep}_F(H)$

Then for **Galois groups**:

- then surjective $\text{Gal}(\text{NNum}_k^\dagger(k)) \twoheadrightarrow \text{Gal}((\text{Num}_k^\dagger(k)/_{-\otimes\mathbb{Q}(1)})^{\text{h}})$ from embedding of subcategory and

$$\text{Gal}((\text{Num}_k^\dagger(k)/_{-\otimes\mathbb{Q}(1)})^{\text{h}}) = \text{Ker}(t : \text{Num}_k^\dagger(k) \rightarrow \mathbb{G}_m)$$

- for super-Tannakian: surjective (from subcategory) $\text{sGal}(\text{NNum}_k(k)) \twoheadrightarrow \text{sGal}((\text{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^{\text{h}})$ and $\text{sGal}((\text{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^{\text{h}}) \simeq \text{Ker}(t : \text{sGal}(\text{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$

- **What is kernel?** $\text{Ker} =$ “truly noncommutative motives”

$$\text{Gal}(\text{NNum}_k^\dagger(k)) \twoheadrightarrow \text{Ker}(t : \text{Num}_k^\dagger(k) \rightarrow \mathbb{G}_m)$$

$$\text{sGal}(\text{NNum}_k(k)) \twoheadrightarrow \text{Ker}(t : \text{sGal}(\text{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look like? examples? general properties?