JEMS

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Noncommutative numerical motives, Tannakian structures, and motivic Galois groups

Abstract. In this article we further the study of noncommutative numerical motives, initiated in [30, 31]. By exploring the change-of-coefficients mechanism, we start by improving some of the main results of [30]. Then, making use of the notion of Schur-finiteness, we prove that the category $\mathrm{NNum}(k)_F$ of noncommutative numerical motives is (neutral) super-Tannakian. As in the commutative world, $\mathrm{NNum}(k)_F$ is not Tannakian. In order to solve this problem we promote periodic cyclic homology to a well-defined symmetric monoidal functor $\overline{HP}_*$ on the category of noncommutative Chow motives. This allows us to introduce the correct noncommutative analogues C_{NC} and D_{NC} of Grothendieck's standard conjectures C and D. Assuming C_{NC}, we prove that $\mathrm{NNum}(k)_F$ can be made into a Tannakian category $\mathrm{NNum}^\dagger(k)_F$ by modifying its symmetry isomorphism constraints. By further assuming D_{NC}, we neutralize the Tannakian category $\mathrm{Num}^\dagger(k)_F$ using $\overline{HP}_*$. Via the (super-)Tannakian formalism, we then obtain well-defined *noncommutative motivic Galois (super-)groups*. Finally, making use of Deligne–Milne's theory of Tate triples, we construct explicit morphisms relating these noncommutative motivic Galois (super-)groups to the classical ones as suggested by Kontsevich.

Keywords. Noncommutative algebraic geometry, noncommutative motives, periodic cyclic homology, Tannakian formalism, motivic Galois groups

Contents

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1. Introduction

Noncommutative motives

Over the past two decades Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Orlov, Van den Bergh, and others, have been promoting a broad noncommutative (algebraic) geometry program where "geometry" is performed directly on dg categories: consult [6, 7, 8, 16, 17, 20, 24, 25, 26, 27]. In this vein, Kontsevich introduced the category $\mathrm{NChow}(k)_F$ of *noncommutative Chow motives* (over a base commutative ring k and with coefficients in a field F; see §3.1). Recently, making use of Hochschild homology, the authors introduced the category $\mathrm{NNum}(k)_F$ of *noncommutative numerical motives* (see §3.2). Under mild assumptions on k and F (see Theorem 5.6) the category $\mathrm{NNum}(k)_F$ is abelian semisimple. The precise relation between $\mathrm{NChow}(k)_\mathbb{Q}$, $\mathrm{NNum}(k)_\mathbb{Q}$, and the classical categories of Chow and numerical motives can be depicted as follows

$$
\begin{array}{ccc}
 & \mathrm{Chow}(k)_\mathbb{Q} & \\
\mathrm{Num}(k)_\mathbb{Q} \quad \mathrm{Chow}(k)_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{\ R\ } & \mathrm{NChow}(k)_\mathbb{Q} \qquad (1.1)\\
\mathrm{Num}(k)_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{\ R_\mathcal{N}\ } & \mathrm{NNum}(k)_\mathbb{Q}
\end{array}
$$

Here, $\mathbb{Q}(1)$ is the Tate motive, $\mathrm{NChow}(k)_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)}$ and $\mathrm{Num}(k)_\mathbb{Q}/_{-\otimes\mathbb{Q}(1)}$ the orbit categories (see §4) associated to the auto-equivalence $-\otimes\mathbb{Q}(1)$, and R and $R_\mathcal{N}$ fully-faithful functors; consult [30] (or the survey [36]) for further details.

Motivating questions

In the commutative world, the category $\mathrm{Num}(k)_F$ of numerical motives is known to be not only abelian semisimple but also (neutral) super-Tannakian. Moreover, assuming the standard conjecture C (or even the sign conjecture C^+), $\mathrm{Num}(k)_F$ can be made into a Tannakian category $\mathrm{Num}^\dagger(k)_F$ by modifying its symmetry isomorphism constraints (see Jannsen [19]). By further assuming the standard conjecture D, the classical Weil cohomologies can be used to neutralize the Tannakian category $\mathrm{Num}^\dagger(k)_F$ (see [2, §6]). As explained in §7, the (super-)Tannakian formalism provides us then with well-defined motivic affine (super-)group schemes $\mathrm{sGal}(\mathrm{Num}(k)_F)$ and $\mathrm{Gal}(\mathrm{Num}^\dagger(k)_F)$ encoding deep arithmetic/geometric properties of smooth projective k-schemes. This circle of results and conjectures in the commutative world leads us naturally to the following questions in the noncommutative world:

Question I. *Is the category* $\mathrm{NNum}(k)_F$ *(neutral) super-Tannakian?*

Question II. *Does the standard conjecture C (or the sign conjecture C^+) admit a non-commutative analogue C_{NC}? Does C_{NC} allow us to make $\mathrm{NNum}(k)_F$ into a Tannakian category $\mathrm{NNum}^\dagger(k)_F$?*

Question III. *Does the standard conjecture D admit a noncommutative analogue D_{NC}? Does D_{NC} allow us to neutralize $\mathrm{NNum}^\dagger(k)_F$?*

Question IV. *Assuming that Questions I, II and III have affirmative answers, how do the noncommutative motivic affine (super-)group schemes hence obtained relate to $\mathrm{sGal}(\mathrm{Num}(k)_F)$ and $\mathrm{Gal}(\mathrm{Num}^\dagger(k)_F)$?*

Statement of results

By exploring the change-of-coefficients mechanism we start by improving the main results of [30] concerning the semisimplicity of the category $\mathrm{NNum}(k)_F$ and the relation between the commutative and the noncommutative world (see Proposition 5.4 and Theorem 5.6). Then, making use of the notion of Schur-finiteness (see §6), we answer Question I affirmatively.

Theorem 1.2. *Assume that F is a field of characteristic zero and that k and F are as in Theorem 5.6. Then the category $\mathrm{NNum}(k)_F$ is super-Tannakian. If F is moreover algebraically closed, then $\mathrm{NNum}(k)_F$ is neutral super-Tannakian.*

Theorem 1.2 (with F algebraically closed) combined with the super-Tannakian formalism gives rise to an affine super-group scheme $\mathrm{sGal}(\mathrm{NNum}(k)_F)$, which we will call the *noncommutative motivic Galois super-group*. Among other consequences, $\mathrm{NNum}(k)_F$ is $\otimes$-equivalent to the category of finite-dimensional F-valued super-representations of $\mathrm{sGal}(\mathrm{NNum}(k)_F)$.

The category $\mathrm{NNum}(k)_F$ is *not* Tannakian since the rank of each one of its objects is not necessarily a nonnegative integer. In order to solve this problem, we start by promoting periodic cyclic homology to a well-defined symmetric monoidal functor $\overline{HP_*}$ on the category of noncommutative Chow motives (see Theorem 9.2). Then, given a smooth and proper dg category $\mathcal{A}$ in the sense of Kontsevich (see §2.1), we formulate the following conjecture:

Noncommutative standard conjecture $C_{NC}(\mathcal{A})$. *The Künneth projectors*

$$\pi_{\mathcal{A}}^\pm : \overline{HP_*}(\mathcal{A}) \twoheadrightarrow \overline{HP_*}^\pm(\mathcal{A}) \hookrightarrow \overline{HP_*}(\mathcal{A})$$

are algebraic, i.e. $\pi_{\mathcal{A}}^\pm = \overline{HP_*}(\underline{\pi}_{\mathcal{A}}^\pm)$, with $\underline{\pi}_{\mathcal{A}}^\pm$ noncommutative correspondences.

As in the commutative world, the noncommutative standard conjecture C_{NC} is stable under tensor products, i.e. $C_{NC}(\mathcal{A}) + C_{NC}(\mathcal{B}) \Rightarrow C_{NC}(\mathcal{A} \otimes_k \mathcal{B})$ (see Proposition 10.2). Its relation to the sign conjecture C^+ (see §10) is the following: given a quasi-compact and quasi-separated k-scheme Z, it is well-known that the derived category $\mathcal{D}_{\mathsf{perf}}(Z)$ of perfect complexes of $\mathcal{O}_Z$-modules admits a natural dg enhancement $\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(Z)$ (consult Lunts–Orlov [29] or [12, Example 4.5]). When Z is moreover smooth and proper, the dg category $\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(Z)$ is smooth and proper in the sense of Kontsevich.

Theorem 1.3. *Let k and F be fields of characteristic zero with k a field extension of F. Then $C^+(Z) \Rightarrow C_{NC}(\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(Z))$.*

Intuitively speaking, Theorem 1.3 shows us that when restricted to the commutative world, the noncommutative standard conjecture C_{NC} is more likely to hold than the sign conjecture C^+ (and therefore than the standard conjecture C). Hence, it answers the first part of Question II affirmatively. Our answer to the second part is the following:

Theorem 1.4. *Assume that k is a field of characteristic zero, and that F is a field extension of k or vice versa. Then, if the noncommutative standard conjecture C_{NC} holds, the category $\mathrm{NNum}(k)_F$ can be made into a Tannakian category $\mathrm{NNum}^\dagger(k)_F$ by modifying its symmetry isomorphism constraints.*

In order to answer Question III, we start by observing that the F-linearized Grothendieck group $K_0(\mathcal{A})_F$ of every smooth and proper dg category $\mathcal{A}$ is endowed with two well-defined equivalence relations: one associated to periodic cyclic homology ($\sim_{\mathrm{hom}}$) and another one associated to numerical equivalence ($\sim_{\mathrm{num}}$) (consult §12 for details). This motivates the following conjecture:

Noncommutative standard conjecture $D_{NC}(\mathcal{A})$. *The following equality holds:*

$$K_0(\mathcal{A})_F/\!\sim_{\mathrm{hom}} = K_0(\mathcal{A})_F/\!\sim_{\mathrm{num}}.$$

Its relation to the standard conjecture D (see §12) is the following.

Theorem 1.5. *Let k and F be fields of characteristic zero with k a field extension of F. Then $D(Z) \Rightarrow D_{NC}(\mathcal{D}_{\mathsf{perf}}^{\mathsf{dg}}(Z))$.*

Similarly to Theorem 1.3, Theorem 1.5 shows us that when restricted to the commutative world, the noncommutative standard conjecture D_{NC} is more likely to hold than the standard conjecture D. Hence, it answers the first part of Question III affirmatively. Our answer to the second part is the following:

Theorem 1.6. *Let k be a field of characteristic zero and F a field extension of k. If the noncommutative standard conjectures C_{NC} and D_{NC} hold, then $\mathrm{NNum}^\dagger(k)_F$ is a neutral Tannakian category. Moreover, an explicit fiber functor neutralizing $\mathrm{NNum}^\dagger(k)_F$ is given by periodic cyclic homology.*

Theorem 1.6 combined with the Tannakian formalism gives rise to an affine group scheme $\mathrm{Gal}(\mathrm{NNum}^\dagger(k)_F)$, which we will call the *noncommutative motivic Galois group*. Since by Theorem 5.6 the category $\mathrm{NNum}^\dagger(k)$ is not only abelian but moreover semi-simple, the affine group scheme $\mathrm{Gal}(\mathrm{NNum}^\dagger(k)_F)$ is *pro-reductive*, i.e. its unipotent radical is trivial (see [2, §2.3.2]). Similarly to the super-Tannakian case, the category $\mathrm{NNum}^\dagger(k)_F$ is $\otimes$-equivalent to the category of finite-dimensional F-valued representations of $\mathrm{Gal}(\mathrm{NNum}^\dagger(k)_F)$.

Finally, making use of all the above results as well as of Deligne–Milne's theory of Tate triples, we answer Question IV as follows:

Theorem 1.7. *Let k be a field of characteristic zero. If the standard conjectures C and D as well as the noncommutative standard conjectures C_{NC} and D_{NC} hold, then there exist well-defined faithfully flat morphisms* (*see* [39, §15.5])

$$\mathrm{sGal}(\mathrm{NNum}(k)_k) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{sGal}(\mathrm{Num}(k)_k) \twoheadrightarrow \mathbb{G}_{\mathrm{m}}), \qquad (1.8)$$

$$\mathrm{Gal}(\mathrm{NNum}^{\dagger}(k)_k) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Gal}(\mathrm{Num}^{\dagger}(k)_k) \twoheadrightarrow \mathbb{G}_{\mathrm{m}}). \qquad (1.9)$$

Here, $\mathbb{G}_{\mathrm{m}}$ denotes the multiplicative group scheme, t is induced by the fully-faithful inclusion of the category of Tate motives, and the affine (super-)group schemes are computed with respect to periodic cyclic homology and de Rham cohomology.

Theorem 1.7 was suggested by Kontsevich [24]. Intuitively speaking, it shows us that the $\otimes$-symmetries of the commutative world which can be lifted to the noncommutative world are precisely those that become trivial when restricted to Tate motives. The difficulty of replacing k by a more general field of coefficients F lies in the lack of an appropriate "noncommutative étale/Betti cohomology".

Conventions. Throughout the article, we will reserve the letter k for the base commutative ring and the letter F for the field of coefficients. The pseudo-abelian envelope construction will be denoted by $(-)^{\natural}$.

2. Dg categories

In this section we collect the notions and results concerning dg categories which are used throughout the article. For further details we invite the reader to consult Keller's ICM address [22]. Let $\mathcal{C}(k)$ the category of (unbounded) cochain complexes of k-modules. A *differential graded* (= *dg*) *category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$. Concretely, the morphisms sets $\mathcal{A}(x, y)$ are complexes of k-modules and the composition operation fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* is a functor which preserves the differential graded structure. The category of small dg categories will be denoted by $\mathrm{dgcat}(k)$.

Let $\mathcal{A}$ be a dg category. Its *opposite* dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and complexes of morphisms given by $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right dg $\mathcal{A}$-module* (or simply an $\mathcal{A}$-module) is a dg functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category of complexes of k-modules. We will denote by $\mathcal{C}(\mathcal{A})$ the category of $\mathcal{A}$-modules and by $\mathcal{D}(\mathcal{A})$ the *derived category* of $\mathcal{A}$, i.e. the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms (consult [22, §3] for details). The full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ of *compact objects* [34, Def. 4.2.7] will be denoted by $\mathcal{D}_c(\mathcal{A})$.

As proved in [38, Thm. 5.3], the category $\mathrm{dgcat}(k)$ carries a Quillen model structure whose weak equivalences are the *derived Morita equivalences*, i.e. the dg functors $\mathcal{A} \to \mathcal{B}$ which induce an equivalence $\mathcal{D}(\mathcal{A}) \overset{\sim}{\to} \mathcal{D}(\mathcal{B})$ on the associated derived categories. The homotopy category hence obtained will be denoted by $\mathrm{Hmo}(k)$.

The tensor product of k-algebras extends naturally to dg categories, giving rise to a symmetric monoidal structure $- \otimes_k -$ on $\mathrm{dgcat}(k)$. The $\otimes$-unit is the dg category $\underline{k}$ with

one object and with k as the dg algebra of endomorphisms (concentrated in degree zero). As explained in [22, §4.3], the tensor product of dg categories can be derived $- \otimes^{\mathbb{L}} -$ thus giving rise to a symmetric monoidal structure on $\mathsf{Hmo}(k)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. A $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a dg functor $\mathcal{A} \otimes^{\mathbb{L}}_k \mathcal{B}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$, or in other words an $(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}}_k \mathcal{B})$-module. Let $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}}_k \mathcal{B})$ spanned by the (cofibrant) $\mathcal{A}$-$\mathcal{B}$-bimodules X such that for every object $x \in \mathcal{A}$ the associated $\mathcal{B}$-module $X(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$.

2.1. Smooth and proper dg categories

Following Kontsevich [24, 25], a dg category $\mathcal{A}$ is called *smooth* if the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A}(-, -) : \mathcal{A} \otimes^{\mathbb{L}}_k \mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k), \quad (x, y) \mapsto \mathcal{A}(x, y),$$

belongs to $\mathcal{D}_c(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}}_k \mathcal{A})$. It is called *proper* if for each ordered pair (x, y) of objects, the complex $\mathcal{A}(x, y)$ of k-modules belongs to $\mathcal{D}_c(k)$. As proved in [12, Thm. 4.8], the smooth and proper dg categories can be conceptually characterized as being precisely the dualizable (or rigid) objects of the symmetric monoidal category $\mathsf{Hmo}(k)$.

3. Noncommutative motives

In this section we recall the construction of the categories of noncommutative Chow and numerical motives. For the mixed case consult [11, 12, 35].

3.1. Noncommutative Chow motives

The rigid symmetric monoidal category $\mathsf{NChow}(k)_F$ of *noncommutative Chow motives* (over a base commutative ring k and with coefficients in a field F) was proposed by Kontsevich [24] and developed in full detail in [37, 38]. It is defined as the pseudo-abelian envelope of the category:

  (i) whose objects are the smooth and proper dg categories;
 (ii) whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the F-linearized Grothendieck group $K_0(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}}_k \mathcal{B})_F$;
(iii) whose composition law is induced by the (derived) tensor product of bimodules.

Its symmetric monoidal structure is induced by the (derived) tensor product of dg categories. In analogy with the commutative world, the morphisms of $\mathsf{NChow}(k)_F$ will be called *noncommutative correspondences*. By definition a *noncommutative Chow motive* is then a pair $(\mathcal{A}, e)$, where $\mathcal{A}$ a smooth and proper dg category and e an idempotent of the F-algebra $K_0(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}}_k \mathcal{A})_F$. When $e = [\mathcal{A}(-, -)]$, we will simply write $\mathcal{A}$ instead of $(\mathcal{A}, [\mathcal{A}(-, -)])$.

3.2. Noncommutative numerical motives

The rigid symmetric monoidal category $\mathrm{NNum}(k)_F$ of *noncommutative numerical motives* (over a base commutative ring k and with coefficients in a field F) was constructed[1] by the authors [30]. Let $(\mathcal{A}, e)$ and $(\mathcal{B}, e')$ be two noncommutative Chow motives and

$$\underline{X} = \left(e \circ \left[\sum_i a_i X_i \right] \circ e' \right) \in \mathrm{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e')),$$

$$\underline{Y} = \left(e' \circ \left[\sum_j b_j Y_j \right] \circ e \right) \in \mathrm{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{B}, e'), (\mathcal{A}, e))$$

two noncommutative correspondences. Recall from the above that X_i is an $\mathcal{A}$-$\mathcal{B}$-bimodule, that Y_j is a $\mathcal{B}$-$\mathcal{A}$-bimodule, and that the sums are indexed by a finite set. The *intersection number* $\langle \underline{X} \cdot \underline{Y} \rangle$ of $\underline{X}$ with $\underline{Y}$ is given by the formula

$$\sum_{i,j} a_i \cdot b_j \cdot [HH(\mathcal{A}; X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j)] \in K_0(k)_F,$$

where $[HH(\mathcal{A}; X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j)]$ denotes the class in $K_0(k)_F$ of the Hochschild homology complex of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-$\mathcal{A}$-bimodule $X_i \otimes_{\mathcal{B}}^{\mathbb{L}} Y_j$. A noncommutative correspondence $\underline{X}$ is called *numerically equivalent to zero* if for every noncommutative correspondence $\underline{Y}$ the intersection number $\langle \underline{X} \cdot \underline{Y} \rangle$ is zero. As proved in [30, Thm. 1.5], the noncommutative correspondences which are numerically equivalent to zero form a $\otimes$-ideal $\mathcal{N}$ of the category $\mathrm{NChow}(k)_F$. Moreover, $\mathcal{N}$ is the largest $\otimes$-ideal of $\mathrm{NChow}(k)_F$ (distinct from the entire category). The category of noncommutative numerical motives $\mathrm{NNum}(k)_F$ is then by definition the pseudo-abelian envelope of the quotient category $\mathrm{NChow}(k)_F/\mathcal{N}$.

4. Orbit categories

In this section (which is of independent interest) we recall the notion of an orbit category and describe its behavior with respect to four distinct operations. In what follows, F will be a field, $(\mathcal{C}, \otimes, \mathbf{1})$ an F-linear, additive, rigid symmetric monoidal category, and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object.

As explained in [37, §7], we can then consider the *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$. It has the same objects as $\mathcal{C}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(X, Y) := \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes j}). \tag{4.1}$$

The composition law is induced by the one on $\mathcal{C}$. By construction, $\mathcal{C}/_{-\otimes\mathcal{O}}$ is F-linear, additive, rigid symmetric monoidal (see [37, Lemma 7.3]), and comes equipped with a canonical projection $\otimes$-functor $\tau : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$. Moreover, τ is endowed with a natural 2-isomorphism $\tau \circ (- \otimes \mathcal{O}) \xrightarrow{\sim} \tau$ and is 2-universal among all such functors.

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[1] Kontsevich [24] had previously introduced a category $\mathrm{NC}_{\mathrm{num}}(k)_F$ of noncommutative numerical motives. However, the authors have recently proved that $\mathrm{NC}_{\mathrm{num}}(k)_F$ and $\mathrm{NNum}(k)_F$ are in fact $\otimes$-equivalent (see [31]).

Change of coefficients

Recall the change-of-coefficients mechanism (see §5).

Lemma 4.2. *Let K be a field extension of F. Then the canonical functor*

$$(\mathcal{C}/_{-\otimes\mathcal{O}}) \otimes_F K \xrightarrow{\sim} (\mathcal{C} \otimes_F K)/_{-\otimes\mathcal{O}}$$

is a $\otimes$-equivalence.

Proof. The proof follows from description (4.1) and from the fact that the functor $-\otimes_F K$, from F-vector spaces to K-vector spaces, commutes with arbitrary sums. $\quad\square$

Pseudo-abelian envelope

The *pseudo-abelian envelope* $\mathcal{C}^\natural$ of $\mathcal{C}$ is defined as follows: the objects are the pairs (X, e) where $X \in \mathcal{C}$ and e is an idempotent of the F-algebra $\mathrm{End}_\mathcal{C}(X)$, and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}^\natural}((X, e), (Y, e')) := e \circ \mathrm{Hom}_\mathcal{C}(X, Y) \circ e'.$$

Composition is naturally induced by $\mathcal{C}$. The symmetric monoidal structure on $\mathcal{C}$ extends naturally to $\mathcal{C}^\natural$ by the formula $(X, e) \otimes (Y, e') := (X \otimes Y, e \otimes e')$.

Lemma 4.3. *We have a fully-faithful, F-linear, additive, $\otimes$-functor*

$$\mathcal{C}^\natural/_{-\otimes\mathcal{O}} \to (\mathcal{C}/_{-\otimes\mathcal{O}})^\natural, \quad (X, e) \mapsto (X, \tau(e)). \tag{4.4}$$

Moreover, the induced functor $\tau^\natural : \mathcal{C}^\natural \to (\mathcal{C}/_{-\otimes\mathcal{O}})^\natural$ factors through (4.4).

Proof. The fact that (4.4) is F-linear, additive, and symmetric monoidal is clear. Let us show that it is moreover fully-faithful. Given objects (X, e) and (Y, e') in $\mathcal{C}^\natural$, we have

$$\mathrm{Hom}_{\mathcal{C}^\natural/_{-\otimes\mathcal{O}}}((X, e), (Y, e')) = \bigoplus_{j\in\mathbb{Z}} e \circ \mathrm{Hom}_\mathcal{C}(X, Y \otimes \mathcal{O}^{\otimes j}) \circ (e' \otimes \mathrm{id}_{\mathcal{O}^{\otimes j}}), \tag{4.5}$$

where $e \otimes \mathrm{id}_{\mathcal{O}^{\otimes j}}$ is an idempotent of $Y \otimes \mathcal{O}^{\otimes j}$. On the other hand,

$$\mathrm{Hom}_{(\mathcal{C}/_{-\otimes\mathcal{O}})^\natural}((X, \tau(e)), (Y, \tau(e'))) = \tau(e) \circ \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_\mathcal{C}(X, Y \otimes \mathcal{O}^{\otimes j}) \circ \tau(e'). \tag{4.6}$$

The composition operation of the orbit category $\mathcal{C}/_{-\otimes\mathcal{O}}$ allows us to conclude that (4.5) = (4.6), which implies that (4.4) is fully-faithful. The fact that $\tau^\natural$ factors through (4.4) is now clear. $\quad\square$

Quotient categories

Lemma 4.7. *Let $H : \mathcal{C}/_{-\otimes\mathcal{O}} \to \mathcal{D}$ be an F-linear, additive $\otimes$-functor. Then we obtain a full, F-linear, additive $\otimes$-functor*

$$(\mathcal{C}/Ker)/_{-\otimes\mathcal{O}} \to (\mathcal{C}/_{-\otimes\mathcal{O}})/Ker(H), \qquad X \mapsto X, \tag{4.8}$$

where Ker denotes the kernel of the composed functor $\mathcal{C} \xrightarrow{\tau} \mathcal{C}/_{-\otimes\mathcal{O}} \xrightarrow{H} \mathcal{D}$. Moreover, the induced functor $\mathcal{C}/Ker \to (\mathcal{C}/_{-\otimes\mathcal{O}})/Ker(H)$ factors through (4.8).

Proof. The fact that the functor (4.8) is F-linear, additive, and symmetric monoidal is clear. In order to show that it is moreover full, note that every morphism $[f] = [\{f_j\}_{j\in\mathbb{Z}}] \in \mathrm{Hom}_{(\mathcal{C}/_{-\otimes\mathcal{O}})/Ker(H)}(X, Y)$, with $\{f_j\}_{j\in\mathbb{Z}} \in \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes \mathcal{O}^{\otimes j})$ admits a canonical lift given by $\tilde{f} = \{[f_j]\}_{j\in\mathbb{Z}} \in \mathrm{Hom}_{(\mathcal{C}/Ker)/_{-\otimes\mathcal{O}}}(X, Y)$, with $\{[f_j]\}_{j\in\mathbb{Z}} \in \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}/Ker}(X, Y \otimes \mathcal{O}^{\otimes j})$. That the induced functor $\mathcal{C}/Ker \to (\mathcal{C}/_{-\otimes\mathcal{O}})/Ker(H)$ factors through (4.8) is now clear. $\square$

Change of symmetry

Recall the general procedure $(-)^{\dagger}$ of changing the symmetry isomorphism constraints of a symmetric monoidal category (see Proposition 11.7).

Lemma 4.9. *Let $H : \mathcal{C} \to \mathrm{sVect}(F)$ be an F-linear $\otimes$-functor with values in the category of finite-dimensional super F-vector spaces. If for every object $X \in \mathcal{C}$, the Künneth projectors $\pi_X^{\pm} : H(X) \twoheadrightarrow H^{\pm}(X) \hookrightarrow H(X)$ can be written as $\pi_X^{\pm} = H(\underline{\pi}_X^{\pm})$ with $\underline{\pi}_X^{\pm} \in \mathrm{End}_{\mathcal{C}}(X)$, then the identity functor is a $\otimes$-equivalence*

$$\mathcal{C}^{\dagger}/_{-\otimes\mathcal{O}} \xrightarrow{\sim} (\mathcal{C}/_{-\otimes\mathcal{O}})^{\dagger}.$$

Proof. Recall from [37, Lemma 7.3] that the symmetry isomorphism constraints $c_{X,Y}$ of the category $\mathcal{C}/_{-\otimes\mathcal{O}}$ are the image of those of $\mathcal{C}$ under the projection functor $\tau : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}$. Similarly, the endomorphisms $\underline{\pi}_X^{+}$ of $\mathcal{C}/_{-\otimes\mathcal{O}}$ are the image of those of $\mathcal{C}$ under τ. The conclusion now follows from the fact that the new symmetry isomorphism constraints are $c_{X,Y}^{\dagger} := c_{X,Y} \circ (e_X \otimes e_Y)$ with $e_X = 2 \cdot \underline{\pi}_X^{+} - \mathrm{id}_X$, and from the fact that the projection functor τ is symmetric monoidal. $\square$

5. Change of coefficients

In this section, we explore the change-of-coefficients mechanism. This will allow us to improve the main results of [30] concerning the semisimplicity of the category $\mathrm{NNum}(k)_F$ and the relation between the commutative and the noncommutative world (see Proposition 5.4 and Theorem 5.6). In what follows, F will be a field, K a field extension of F, and $(\mathcal{C}, \otimes, \mathbf{1})$ an F-linear, additive, rigid symmetric monoidal category such

that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq F$. These data allow us to consider a new category $\mathcal{C} \otimes_F K$. It has the same objects as $\mathcal{C}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{C} \otimes_F K}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_F K.$$

By construction, it is K-linear, additive, rigid symmetric monoidal, and such that $\mathrm{End}_{\mathcal{C} \otimes_F K}(\mathbf{1}) \simeq F \otimes_F K \simeq K$. Moreover, it comes equipped with a canonical F-linear symmetric monoidal functor $\mathcal{C} \to \mathcal{C} \otimes_F K$.

Quotient categories

Recall from [3, Lemma 7.1.1] that the formula

$$\mathcal{N}_{\mathcal{C}}(X, Y) := \{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \,|\, \forall g \in \mathrm{Hom}_{\mathcal{C}}(Y, X),\ \mathrm{tr}(g \circ f) = 0\}$$

defines a $\otimes$-ideal $\mathcal{N}_{\mathcal{C}}$ of $\mathcal{C}$, where tr stands for the categorical trace. Moreover, $\mathcal{N}_{\mathcal{C}}$ can be characterized as the largest $\otimes$-ideal of $\mathcal{C}$ (distinct from the entire category; see [3, Prop. 7.1.4]). As proved in [10, Prop. 1.4.1], the change-of-coefficients mechanism is well-behaved with respect to this $\otimes$-ideal, i.e. the canonical functor

$$(\mathcal{C}/\mathcal{N}_{\mathcal{C}}) \otimes_F K \xrightarrow{\sim} (\mathcal{C} \otimes_F K)/\mathcal{N}_{\mathcal{C} \otimes_F K} \tag{5.1}$$

is an additive $\otimes$-equivalence of categories.

Motives versus noncommutative motives

As explained in §3.1, the category $\mathrm{NChow}(k)_F$ of noncommutative Chow motives is F-linear, additive, and rigid symmetric monoidal. When k is furthermore local (or more generally when $K_0(k) = \mathbb{Z}$), we have

$$\mathrm{End}_{\mathrm{NChow}(k)_F}(\underline{k}) \simeq K_0(\underline{k}^{\mathrm{op}} \otimes_k^{\mathbb{L}} \underline{k})_F \simeq K_0(k)_F \simeq F.$$

By construction we then observe that the canonical functor

$$(\mathrm{NChow}(k)_F \otimes_F K)^{\natural} \xrightarrow{\sim} \mathrm{NChow}(k)_K \tag{5.2}$$

is an additive $\otimes$-equivalence of categories. By combining (5.1) with (5.2), we obtain the additive $\otimes$-equivalence of categories

$$(\mathrm{NNum}(k)_F \otimes_F K)^{\natural} \xrightarrow{\sim} \mathrm{NNum}(k)_K. \tag{5.3}$$

Proposition 5.4. *Let k and F be fields with F of characteristic zero. Then by applying the change-of-coefficients construction $-\otimes_{\mathbb{Q}} F$ to (1.1) we obtain the following diagram of F-linear, additive, symmetric monoidal functors:*

$$
\begin{array}{ccccc}
& & \mathrm{Chow}(k)_F & & \\
& & {\scriptstyle\tau}\downarrow & & \\
\mathrm{Num}(k)_F & & \mathrm{Chow}(k)_F/_{-\otimes\mathbb{Q}(1)} \xrightarrow{\quad R \quad} & \mathrm{NChow}(k)_F & \\
{\scriptstyle\tau}\downarrow & & & & \\
\mathrm{Num}(k)_F/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{\quad R_{\mathcal{N}} \quad} & & \mathrm{NNum}(k)_F &
\end{array}
\qquad (5.5)
$$

where the functors τ are faithful and the functors R and $R_{\mathcal{N}}$ are fully-faithful.

Proof. As explained in [2, §4.2.2] we have $\otimes$-equivalences

$$(\mathrm{Chow}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural} \xrightarrow{\sim} \mathrm{Chow}(k)_F, \qquad (\mathrm{Num}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural} \xrightarrow{\sim} \mathrm{Num}(k)_F.$$

By combining Lemmas 4.2 and 4.3 we obtain induced fully-faithful functors

$$(\mathrm{Chow}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural} \to (\mathrm{Chow}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural}/_{-\otimes\mathbb{Q}(1)} \to ((\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}) \otimes_{\mathbb{Q}} F)^{\natural},$$

$$(\mathrm{Num}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural} \to (\mathrm{Num}(k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} F)^{\natural}/_{-\otimes\mathbb{Q}(1)} \to ((\mathrm{Num}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}) \otimes_{\mathbb{Q}} F)^{\natural}.$$

Hence, the conclusion follows from the above $\otimes$-equivalences (5.2) and (5.3). $\qquad\square$

Semisimplicity

In this subsection we switch the roles of F and K, i.e. we assume that F is a field extension of K.

Theorem 5.6. *Assume one of the following two conditions holds:*

(i) *The base ring k is local (or more generally $K_0(k) = \mathbb{Z}$) and F is a k-algebra; a large class of examples is given by taking $k = \mathbb{Z}$ and F an arbitrary field.*

(ii) *The base ring k and the field F are two field extensions of a (nontrivial) field K; a large classe of examples is given by taking $K = F = \mathbb{Q}$ and k a field of characteristic zero, or $K = \mathbb{Q}$, $F = \overline{\mathbb{Q}}$, and k a field of characteristic zero.*

Then the category $\mathrm{NNum}(k)_F$ is abelian semisimple. Moreover, if $\mathcal{J}$ is a $\otimes$-ideal in $\mathrm{NChow}(k)_F$ for which the pseudo-abelian envelope of the quotient category $\mathrm{NChow}(k)_F/\mathcal{J}$ is abelian semisimple, then $\mathcal{J}$ agrees with $\mathcal{N}$.

Remark 5.7. Theorem 5.6 extends the original semisimplicity result [30, Thm. 1.9]. The latter is obtained from the former by taking $K = F$.

Proof. Condition (i) is the same as the one of [30, Thm. 1.9]. Hence, let us assume (ii) holds. Since k is a field extension of K we conclude by [30, Thm. 1.9] that the category $\mathrm{NNum}(k)_K$ is abelian semisimple. By hypothesis, F is also a field extension of K and so

by (5.3) (with F and K switched) we obtain an equivalence of categories

$$(\mathrm{NNum}(k)_K \otimes_K F)^{\natural} \xrightarrow{\sim} \mathrm{NNum}(k)_F.$$

In order to prove that $\mathrm{NNum}(k)_F$ is abelian semisimple it suffices then to show that for every object $N \in \mathrm{NNum}(k)K \otimes_K F$, the F-algebra $\mathrm{End}_{\mathrm{NNum}(k)_K \otimes_K F}(N)$ of endomorphisms is finite-dimensional and semisimple (see [19, Lemma 2]). Note that we have a natural isomorphism

$$\mathrm{End}_{\mathrm{NNum}(k)_K \otimes_K F}(N) \simeq \mathrm{End}_{\mathrm{NNum}(k)_K}(N) \otimes_K F. \tag{5.8}$$

The category $\mathrm{NNum}(k)_K$ is abelian semisimple and so the K-algebra $\mathrm{End}_{\mathrm{NNum}(k)_K}(N)$ is finite-dimensional and its Jacobson radical is trivial. By (5.8), the F-algebra $\mathrm{End}_{\mathrm{NNum}(k)_K \otimes_K F}(N)$ is also finite-dimensional. Moreover, since the Jacobson radical of $\mathrm{End}_{\mathrm{NNum}(k)_K \otimes_K F}(N)$ is obtained from the one of $\mathrm{End}_{\mathrm{NNum}(k)_K}(N)$ by base change, it is also trivial (see [3, Prop. 4.1.1]). This implies that the F-algebra $\mathrm{End}_{\mathrm{NNum}(k)_K \otimes_K F}(N)$ is semisimple and so the category $\mathrm{NNum}(k)_F$ is abelian semisimple. The "moreover" claim follows from [3, Prop. 7.1.4 c)]. □

6. Schur-finiteness

In what follows, F will be a field of characteristic zero. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be an F-linear, idempotent complete, symmetric monoidal category. As explained in [1, §3.1], given an object $X \in \mathcal{C}$ and an integer $n \geq 1$, the symmetric group S_n acts on $X^{\otimes n}$ by permutation of its factors. The isomorphism classes V_λ of the irreducible $\mathbb{Q}$-linear representations of S_n are in canonical bijection with the partitions λ of n. As a consequence, the group ring $\mathbb{Q}[S_n]$ can be written as $\prod_{\lambda, |\lambda|=n} \mathrm{End}_{\mathbb{Q}} V_\lambda$. Let c_λ be the idempotent of $\mathbb{Q}[S_n]$ defining the representation V_λ. The *Schur functor* S_λ is then defined by the

$$S_\lambda : \mathcal{C} \to \mathcal{C}, \quad X \mapsto S_\lambda(X) := c_\lambda(X^{\otimes n}).$$

Note that for $\lambda = (n)$ the Schur functor $S_{(n)}$ is the symmetric power functor, while for $\lambda = (1, \ldots, 1)$ the Schur functor $S_{(1,\ldots,1)}$ is the exterior power functor. An object $X \in \mathcal{C}$ is called *Schur-finite* if there exists an integer $n \geq 1$ and a partition λ of n such that $S_\lambda(X) = 0$. The category $\mathcal{C}$ is called *Schur-finite* if all its objects are Schur-finite.

Lemma 6.1. *Let $L : \mathcal{C}_1 \to \mathcal{C}_2$ be an F-linear symmetric monoidal functor.*

(i) *If $X \in \mathcal{C}_1$ is Schur-finite, then $L(X)$ is Schur-finite.*

(ii) *When L is moreover faithful, then the converse holds: if $L(X)$ is Schur-finite, then X is Schur-finite.*

Proof. An easy exercise left for the reader. □

7. Tannakian formalism

In this section we collect some standard notions and results of the (super-)Tannakian formalism. In what follows, F will be a field and $(\mathcal{C}, \otimes, \mathbf{1})$ an F-linear, abelian, rigid symmetric monoidal category such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq F$.

Recall from Deligne [13] that a K-*valued fiber functor on* $\mathcal{C}$ (with K a field extension of F) is an exact faithful $\otimes$-functor $\omega : \mathcal{C} \to \mathrm{Vect}(K)$ with values in the category of finite-dimensional K-vector spaces. The category $\mathcal{C}$ is called *Tannakian* if it admits a K-valued fiber functor. It is called *neutral Tannakian* if it admits an F-valued fiber functor. In this latter case, the fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}(F)$ gives rise to a $\otimes$-equivalence between $\mathcal{C}$ and the category $\mathrm{Rep}_F(\mathrm{Gal}(\mathcal{C}))$ of finite-dimensional F-valued representations of the affine group scheme $\mathrm{Gal}(\mathcal{C}) := \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Theorem 7.1 (Deligne's intrinsic characterization [13]). *Let F be a field of characteristic zero. Then $\mathcal{C}$ is Tannakian if and only if the rank $\mathrm{rk}(N) := \mathrm{tr}(\mathrm{id}_N)$ of each one of its objects N is a nonnegative integer.*

Following Deligne [14], a K-*valued super-fiber functor on* $\mathcal{C}$ (with K a field extension of F) is an exact faithful $\otimes$-functor $\omega : \mathcal{C} \to \mathrm{sVect}(K)$ with values in the category of finite-dimensional super K-vector spaces. The category $\mathcal{C}$ is called *super-Tannakian* if it admits a K-valued super-fiber functor. It is called *neutral super-Tannakian* if it admits an F-valued super-fiber functor. In this latter case, the super-fiber functor $\omega : \mathcal{C} \to \mathrm{sVect}(F)$ gives rise to a $\otimes$-equivalence between $\mathcal{C}$ and the category $\mathrm{Rep}_F(\mathrm{sGal}(\mathcal{C}), \epsilon)$ of finite-dimensional F-valued super-representations of the affine super-group scheme $\mathrm{sGal}(\mathcal{C}) := \underline{\mathrm{Aut}}^{\otimes}(\omega)$, where ϵ is the parity automorphism implementing the super-symmetry of ω.

Theorem 7.2 (Deligne's intrinsic characterization [14]). *Let F be a field of characteristic zero. Then $\mathcal{C}$ is super-Tannakian if and only if is Schur-finite (see §6). If F is moreover algebraically closed, then $\mathcal{C}$ is neutral super-Tannakian if and only if it is Schur-finite.*

8. Super-Tannakian structure

In this section we prove Theorem 1.2.

Proposition 8.1. *Let k and F be as in Theorem 5.6 with F of characteristic zero. Then the category $\mathrm{NNum}(k)_F$ is Schur-finite.*

Proof. Note first that by construction the category $\mathrm{NNum}(k)_F$ is F-linear, idempotent complete, and symmetric monoidal. Let us assume first that k and F satisfy condition (i) of Theorem 5.6. Then, as explained in the proof of [30, Thm. 1.9], Hochschild homology (HH) gives rise to an F-linear symmetric monoidal functor

$$\overline{HH} : \mathrm{NChow}(k)_F \to \mathcal{D}_c(F). \tag{8.2}$$

Let us denote by $\mathrm{Ker}(\overline{HH})$ the associated kernel. Since this is a $\otimes$-ideal of $\mathrm{NChow}(k)_F$, the induced functor

$$(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HH}))^{\natural} \to \mathcal{D}_c(F)$$

is not only F-linear and faithful but moreover symmetric monoidal. Note that the category $\mathcal{D}_c(F)$ can be naturally identified with the category of $\mathbb{Z}$-graded F-vector spaces $\{V_n\}_{n\in\mathbb{Z}}$ such that $\dim(\bigoplus_n V_n) < \infty$. As a consequence, $\mathcal{D}_c(F)$ is Schur-finite. By Lemma 6.1(ii) we then conclude that $(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HH}))^\natural$ is Schur-finite. Now, recall from §3.2 that the ideal $\mathcal{N}$ is the largest $\otimes$-ideal of $\mathrm{NChow}(k)_F$ (distinct from the entire category). The inclusion $\mathrm{Ker}(\overline{HH}) \subset \mathcal{N}$ of $\otimes$-ideals gives rise to an F-linear symmetric monoidal functor

$$(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HH}))^\natural \to (\mathrm{NChow}(k)_F/\mathcal{N})^\natural =: \mathrm{NNum}(k)_F. \qquad (8.3)$$

By combining Lemma 6.1(i) with the fact that the category $(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HH}))^\natural$ is Schur-finite, we conclude that all noncommutative numerical motives in the image of the functor (8.3) are Schur-finite. Finally, since every noncommutative numerical motive is a direct factor of one of these and Schur-finiteness is clearly stable under direct factors, we conclude that the category $\mathrm{NNum}(k)_F$ is Schur-finite.

Let us now assume that k and F satisfy condition (ii) of Theorem 5.6. If $K = F$, then k is a field extension of F and so the proof is similar to the one above: the functor (8.2) takes values not in $\mathcal{D}_c(F)$ but in $\mathcal{D}_c(k)$, but since $\mathcal{D}_c(k)$ is also Schur-finite the same reasoning applies. Now, let us assume that k and F are two field extensions of a (nontrivial) field K. Since F is of characteristic zero, so is K. In this case, k is a field extension of K, and so the preceding argument shows that $\mathrm{NNum}(k)_K$ is Schur-finite. As explained in §5, we have a canonical K-linear symmetric monoidal functor $\mathrm{NNum}(k)_K \to \mathrm{NNum}(k)_K \otimes_K F$. By construction the categories $\mathrm{NNum}(k)_K$ and $\mathrm{NNum}(k)_K \otimes_K F$ have the same objects and so Lemma 6.1(i), combined with the fact that Schur-finiteness is stable under direct factors, implies that $(\mathrm{Num}(k)_K \otimes_K F)^\natural$ is also Schur-finite. Finally, the $\otimes$-equivalence (5.3) (with F and K switched) shows that $\mathrm{NNum}(k)_F$ is Schur-finite. $\qquad\square$

Proposition 8.1 implies a similar result in the commutative world.

Corollary 8.4. *Let k and F be fields of characteristic zero. Then the category $\mathrm{Num}(k)_F$ is Schur-finite.*

Proof. Recall from diagram (5.5) the following composition of faithful, F-linear, additive, symmetric monoidal functors:

$$\mathrm{Num}(k)_F \to \mathrm{Num}(k)_F/_{-\otimes\mathbb{Q}(1)} \xrightarrow{R_\mathcal{N}} \mathrm{NNum}(k)_F.$$

Since k and F are of characteristic zero, they satisfy condition (ii) of Theorem 5.6 (with $K = \mathbb{Q}$). Then Proposition 8.1 combined with Lemma 6.1(ii) shows that $\mathrm{Num}(k)_F$ is Schur-finite. $\qquad\square$

Proof of Theorem 1.2. Note first that by construction the category $\mathrm{NNum}(k)_F$ is F-linear, additive, and rigid symmetric monoidal. Its $\otimes$-unit is the noncommutative Chow motive $\underline{k}$. Since k and F are as in Theorem 5.6, we have

$$\mathrm{End}_{\mathrm{NNum}(k)_F}(\underline{k}) \simeq K_0(k)_F \simeq F.$$

Moreover, due to Theorem 5.6 the category $\mathrm{NNum}(k)_F$ is also abelian (semisimple). By Deligne's intrinsic characterization (Theorem 7.2) it then suffices to show that $\mathrm{NNum}(k)_F$ is Schur-finite. This is the content of Proposition 8.1. $\qquad\square$

9. Periodic cyclic homology

In this section we prove that periodic cyclic homology (*HP*) gives rise to a well-defined symmetric monoidal functor on the category of noncommutative Chow motives (see Theorem 9.2). In what follows, k will be a field.

Following Kassel [21, §1], a *mixed complex* (M, b, B) is a $\mathbb{Z}$-graded k-vector space $\{M_n\}_{n \in \mathbb{Z}}$ endowed with a degree $+1$ endomorphism b and a degree -1 endomorphism B satisfying the relations $b^2 = B^2 = Bb + bB = 0$. Equivalently, a mixed complex is a right dg module over the dg algebra $\Lambda := k[\epsilon]/\epsilon^2$, where ϵ is of degree -1 and $d(\epsilon) = 0$. Recall from [12, Example 7.10] the construction of the mixed complex functor $C : \mathrm{dgcat}(k) \to \mathcal{D}(\Lambda)$ with values in the derived category of Λ. As explained by Kassel [21, p. 210], there is a well-defined 2-*periodization functor* sending a mixed complex (M, b, B) to the $\mathbb{Z}/2$-graded complex of k-vector spaces

$$\prod_{n \text{ even}} M_n \underset{b+B}{\overset{b+B}{\rightleftarrows}} \prod_{n \text{ odd}} M_n.$$

This functor preserves weak equivalences and when combined with C gives rise to periodic cyclic homology

$$HP : \mathrm{dgcat}(k) \xrightarrow{C} \mathcal{D}(\Lambda) \to \mathcal{D}_{\mathbb{Z}/2}(k). \tag{9.1}$$

Here, $\mathcal{D}_{\mathbb{Z}/2}(k)$ stands for the derived category of $\mathbb{Z}/2$-graded complexes.

Theorem 9.2. *When F is a field extension of k, the functor* (9.1) *gives rise to an F-linear symmetric monoidal functor*

$$\overline{HP_*} : \mathrm{NChow}(k)_F \to \mathrm{sVect}(F) \tag{9.3}$$

with values in the category of finite-dimensional super F-vector spaces. On the other hand, when k is a field extension of F, the functor (9.1) *gives rise to an F-linear symmetric monoidal functor*

$$\overline{HP_*} : \mathrm{NChow}(k)_F \to \mathrm{sVect}(k) \tag{9.4}$$

with values in the category of finite-dimensional super k-vector spaces.

Proof. As explained in [12, Example 7.10], the mixed complex functor C is symmetric monoidal. On the other hand, the 2-periodization functor is not symmetric monoidal. This is due to the fact that it uses infinite products and these do not commute with the tensor product. Nevertheless, as explained in [21, p. 210], the 2-periodization functor is lax symmetric monoidal. Note that since by hypothesis k is a field, the category $\mathcal{D}_{\mathbb{Z}/2}(k)$ is $\otimes$-equivalent to the category $\mathrm{SVect}(k)$ of super k-vector spaces. Hence, the functor (9.1) gives rise to a well-defined lax symmetric monoidal functor

$$HP_* : \mathrm{dgcat}(k) \xrightarrow{C} \mathcal{D}(\Lambda) \to \mathcal{D}_{\mathbb{Z}/2}(k) \simeq \mathrm{SVect}(k). \tag{9.5}$$

Now, recall from [38, §5] the construction of the additive category $\mathrm{Hmo}_0(k)$. Its objects are the small dg categories, its morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the Grothendieck group $K_0 \mathrm{rep}(\mathcal{A}, \mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ (see §2), and its composition law is induced by the (derived) tensor product of bimodules. The (derived) tensor product

of dg categories endows $\mathsf{Hmo}_0(k)$ with a symmetric monoidal structure. Moreover, there is a natural symmetric monoidal functor $\mathcal{U} : \mathsf{dgcat}(k) \to \mathsf{Hmo}_0(k)$ which can be characterized as the *universal additive invariant* (consult [36, 38], [22, §5.1] for details). The above functor (9.5) is an example of an *additive invariant*, i.e. it inverts derived Morita equivalences (see §2) and maps semiorthogonal decompositions in the sense of Bondal-Orlov [9] into direct sums. Hence, by the universal property of $\mathcal{U}$, the additive invariant (9.5) gives rise to a well-defined lax symmetric monoidal functor

$$\overline{HP_*} : \mathsf{Hmo}_0(k) \to \mathsf{SVect}(k) \tag{9.6}$$

such that $\overline{HP_*} \circ \mathcal{U} = HP_*$. Now, let us denote by $\mathsf{Hmo}_0(k)^{\mathrm{sp}} \subset \mathsf{Hmo}_0(k)$ the full subcategory of smooth and proper dg categories in the sense of Kontsevich (see §2.1). Given smooth and proper dg categories $\mathcal{A}$ and $\mathcal{B}$, there is a natural equivalence of categories $\mathsf{rep}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B})$. As a consequence,

$$\mathsf{Hom}_{\mathsf{Hmo}_0(k)^{\mathrm{sp}}}(\mathcal{A}, \mathcal{B}) := K_0\mathsf{rep}(\mathcal{A}, \mathcal{B}) \simeq K_0(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B}).$$

This shows that the category $\mathsf{NChow}(k)_F$ can be obtained from $\mathsf{Hmo}_0(k)^{\mathrm{sp}}$ by first tensoring each abelian group of morphisms with the field F and then passing to the associated pseudo-abelian envelope. Schematically, we have the composition

$$\mathsf{Hmo}_0(k)^{\mathrm{sp}} \xrightarrow{(-)_F} \mathsf{Hmo}_0(k)^{\mathrm{sp}}_F \xrightarrow{(-)^\natural} \mathsf{NChow}(k)_F. \tag{9.7}$$

By Proposition 9.9(ii), the restriction of (9.6) to $\mathsf{Hmo}_0(k)^{\mathrm{sp}}$ gives rise to a lax symmetric monoidal functor

$$\overline{HP_*} : \mathsf{Hmo}_0(k)^{\mathrm{sp}} \to \mathsf{sVect}(k). \tag{9.8}$$

Moreover, conditions (i) and (ii) of Proposition 9.9 are precisely what Emmanouil named "property (II)" in [18, p. 211]. As a consequence, [18, Thm. 4.2] implies that the functor (9.8) is in fact symmetric monoidal.

We now have all the ingredients needed for the description of the functors (9.3) and (9.4). Let us assume first that F is a field extension of k. Then we have an induced extension of scalars functor $\mathsf{sVect}(k) \to \mathsf{sVect}(F)$. This functor is symmetric monoidal and the category $\mathsf{sVect}(F)$ is clearly idempotent complete. Using the description of $\mathsf{NChow}(k)_F$ given by (9.7), we conclude that (9.8) extends to $\mathsf{NChow}(k)_F$, thus giving rise to the F-linear symmetric monoidal functor (9.3). Now, let us assume that k is a field extension of F. In this case the category $\mathsf{sVect}(k)$ is already F-linear. Hence, using the description of $\mathsf{NChow}(k)_F$ in (9.7), we conclude that (9.8) gives rise to the induced F-linear symmetric monoidal functor (9.4). $\qquad\square$

Proposition 9.9. *Let $\mathcal{A}$ be a smooth and proper dg category. Then:*

(i) *The inverse system $(HC(\mathcal{A})[-2m], S)_m$ of cyclic homology k-vector spaces (see [28, §2.2]) satisfies the Mittag-Leffler condition.*

(ii) *The periodic cyclic homology k-vector spaces $HP_n(\mathcal{A})$ are finite-dimensional.*

Proof. An easy exercise left for the reader. $\qquad\square$

10. Noncommutative standard conjecture C_{NC}

In this section we start by recalling the standard conjecture C and the sign conjecture C^+. Then, we prove that the noncommutative standard conjecture C_{NC} is stable under tensor products (Proposition 10.2), and finally we prove Theorem 1.3. In what follows, Z will be a smooth projective k-scheme over a field k of characteristic zero.

Recall from [2, §3.4] that with respect to de Rham cohomology, we have the Künneth projectors

$$\pi_Z^n : H_{dR}^*(Z) \twoheadrightarrow H_{dR}^n(Z) \hookrightarrow H_{dR}^*(Z), \quad 0 \le n \le 2\dim(Z).$$

Standard conjecture $C(Z)$ (see [2, §5.1.1.1]). *The Künneth projectors π_Z^n are algebraic, i.e. they can be written as $\pi_Z^n = H_{dR}^*(\underline{\pi}_Z^n)$, with $\underline{\pi}_Z^n$ algebraic correspondences.*

Sign conjecture $C^+(Z)$ (see [2, §5.1.3]). *The Künneth projectors $\pi_Z^+ := \sum_{n=0}^{\dim(Z)} \pi_Z^{2n}$ (and hence $\pi_Z^- := \sum_{n=1}^{\dim(Z)} \pi_Z^{2n-1}$) are algebraic, i.e. they can be written as $\pi_Z^+ = H_{dR}^*(\underline{\pi}_Z^+)$, with $\underline{\pi}_Z^+$ algebraic correspondences.*

Remark 10.1. Clearly $C(Z) \Rightarrow C^+(Z)$. Recall also from [2, §5.1.1.2] that $C(Z)$ and $C^+(Z)$ could equivalently be formulated using any other classical Weil cohomology theory.

Proposition 10.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two smooth and proper dg categories* (*see §2.1*). *Then*

$$C_{NC}(\mathcal{A}) + C_{NC}(\mathcal{B}) \Rightarrow C_{NC}(\mathcal{A} \otimes_k \mathcal{B}). \tag{10.3}$$

Proof. Recall from Theorem 9.2 that we have a well-defined F-linear symmetric monoidal functor

$$\overline{HP_*} : \mathrm{NChow}(k)_F \to \mathrm{sVect}(K). \tag{10.4}$$

The field K is equal to F when F is a field extension of k and is equal to k when k is a field extension of F. Let us denote by $-\widehat{\otimes}-$ the symmetric monoidal structure on $\mathrm{sVect}(K)$. Then we have the following equalities between the Künneth projectors:

$$\pi_{\mathcal{A} \otimes_k \mathcal{B}}^+ = \pi_{\mathcal{A}}^+ \widehat{\otimes} \pi_{\mathcal{B}}^+ + \pi_{\mathcal{A}}^+ \widehat{\otimes} \pi_{\mathcal{B}}^-, \quad \pi_{\mathcal{A} \otimes_k \mathcal{B}}^- = \pi_{\mathcal{A}}^+ \widehat{\otimes} \pi_{\mathcal{B}}^- + \pi_{\mathcal{A}}^- \widehat{\otimes} \pi_{\mathcal{B}}^-.$$

Since (10.4) is symmetric monoidal and by hypothesis we have noncommutative correspondences $\underline{\pi}_{\mathcal{A}}^{\pm}$ and $\underline{\pi}_{\mathcal{B}}^{\pm}$ such that $\pi_{\mathcal{A}}^{\pm} = \overline{HP_*}(\underline{\pi}_{\mathcal{A}}^{\pm})$ and $\pi_{\mathcal{B}}^{\pm} = \overline{HP_*}(\underline{\pi}_{\mathcal{B}}^{\pm})$, the noncommutative standard conjecture $C_{NC}(\mathcal{A} \otimes_k \mathcal{B})$ also holds. As a consequence, (10.3) holds. $\square$

Proof of Theorem 1.3. As explained in [2, §4.2.5], de Rham cohomology H_{dR}^* (considered as a Weil cohomology theory) gives rise to a symmetric monoidal functor

$$\overline{H_{dR}^*} : \mathrm{Chow}(k)_F \to \mathrm{GrVect}(k) \tag{10.5}$$

with values in the category of finite-dimensional $\mathbb{Z}$-graded k-vector spaces. On the other hand, as explained in Theorem 9.2, periodic cyclic homology gives rise to a well-defined symmetric monoidal functor

$$\overline{HP_*} : \mathrm{Chow}(k)_F \to \mathrm{sVect}(k). \tag{10.6}$$

Recall from diagram (5.5) the sequence of functors

$$\mathrm{Chow}(k)_F \xrightarrow{\tau} \mathrm{Chow}(k)_F/_{-\otimes \mathbb{Q}(1)} \xrightarrow{R} \mathrm{NChow}(k)_F. \tag{10.7}$$

By combining [37, Thm. 1.1] with the change-of-coefficients mechanism we conclude that the image of Z under the above composition (10.7) identifies naturally with the non-commutative Chow motive $\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z)$. Now, recall from Keller [23] that $\overline{HP_*}(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z)) \simeq HP_*(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z))$ agrees with the periodic cyclic homology $HP_*(Z)$ of the k-scheme Z in the sense of Weibel [40]. Since k is a field of characteristic zero and Z is smooth, the Hochschild–Kostant–Rosenberg theorem [41] furnishes us with the isomorphisms

$$HP_*^+(Z) \simeq \bigoplus_{n \text{ even}} H_{dR}^n(Z) \qquad HP_*^-(Z) \simeq \bigoplus_{n \text{ odd}} H_{dR}^n(Z).$$

These facts show that the composition of (10.6) with (10.7) is the functor

$$s\overline{H_{dR}^*} : \mathrm{Chow}(k)_F \to \mathrm{sVect}(k), \quad Z \mapsto \Big(\bigoplus_{n \text{ even}} H_{dR}^n(Z), \bigoplus_{n \text{ odd}} H_{dR}^n(Z) \Big), \tag{10.8}$$

naturally associated to (10.5). As a consequence, the Künneth projectors $\pi_Z^{\pm}$ of the sign conjecture $C^+(Z)$ agree with the Künneth projectors $\pi_{\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z)}^{\pm}$ of the noncommutative standard conjecture $C_{NC}(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z))$. Now, if by hypothesis the sign conjecture $C^+(Z)$ holds, there exist algebraic correspondences $\underline{\pi}_Z^{\pm}$ realizing the Künneth projectors $\pi_Z^{\pm}$. Hence, by taking for noncommutative correspondences $\pi_{\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z)}^{\pm}$ the image of $\underline{\pi}_Z^{\pm}$ under the above functor (10.7) we conclude that the noncommutative standard conjecture $C_{NC}(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(Z))$ also holds. $\square$

11. Tannakian structure

In this section we prove Theorem 1.4. In what follows, k will be a field of characteristic zero.

Proposition 11.1. *Assume that F is a field extension of k or vice versa and that the non-commutative standard conjecture $C_{NC}(\mathcal{A})$ holds for a smooth and proper dg category $\mathcal{A}$. Then, given any noncommutative Chow motive of shape $(\mathcal{A}, e)$ (see §3.1), the kernel of the induced surjective ring homomorphism*

$$\mathrm{End}_{\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})}((\mathcal{A}, e)) \twoheadrightarrow \mathrm{End}_{\mathrm{NChow}(k)_F/\mathcal{N}}((\mathcal{A}, e))$$

is a nilpotent ideal.

Proof. Let us consider first the case where F is a field extension of k. By Theorem 9.2, we have an F-linear symmetric monoidal functor

$$\overline{HP_*} : \mathrm{NChow}(k)_F \to \mathrm{sVect}(F). \tag{11.2}$$

The associated kernel $\text{Ker}(\overline{HP_*})$ is a $\otimes$-ideal of $\text{NChow}(k)_F$ and so we obtain an induced functor

$$\text{NChow}(k)_F/\text{Ker}(\overline{HP_*}) \to \text{sVect}(F), \qquad (11.3)$$

which is not only F-linear and faithful but moreover symmetric monoidal. Recall from §3.2 that the ideal $\mathcal{N}$ is the largest $\otimes$-ideal of $\text{NChow}(k)_F$ (distinct from the entire category). Hence, the induced functor

$$\text{NChow}(k)_F/\text{Ker}(\overline{HP_*}) \to \text{NChow}(k)_F/\mathcal{N}$$

is not only F-linear and symmetric monoidal but moreover full and (essentially) surjective. Now, observe that if by hypothesis the noncommutative standard conjecture $C_{NC}(\mathcal{A})$ holds for a smooth and proper dg category $\mathcal{A}$, then the Künneth projectors

$$\pi^{\pm}_{(\mathcal{A},e)} : \overline{HP_*}((\mathcal{A}, e)) \twoheadrightarrow \overline{HP_*}^{\pm}((\mathcal{A}, e)) \hookrightarrow \overline{HP_*}((\mathcal{A}, e)) \qquad (11.4)$$

associated to a noncommutative Chow motive $(\mathcal{A}, e)$ are also algebraic. Simply take for $\underline{\pi}^{\pm}_{(\mathcal{A},e)}$ the noncommutative correspondence $e \circ \underline{\pi}^{\pm}_{\mathcal{A}} \circ e$. Let $\underline{X} \in \text{End}_{\text{NChow}(k)_F}((\mathcal{A}, e))$ be a noncommutative correspondence. We need to show that if $\underline{X}$ becomes trivial in $\text{NChow}(k)_F/\mathcal{N}$, then it is nilpotent in $\text{NChow}(k)_F/\text{Ker}(\overline{HP_*})$. As explained above, the Künneth projectors (11.4) can be written as $\pi^{\pm}_{(\mathcal{A},e)} = \overline{HP_*}(\underline{\pi}^{\pm}_{(\mathcal{A},e)})$, with $\underline{\pi}^{\pm}_{(\mathcal{A},e)} \in \text{End}_{\text{NChow}(k)_F}((\mathcal{A}, e))$. If by hypothesis $\underline{X}$ becomes trivial in $\text{NChow}(k)_F/\mathcal{N}$, then by definition of $\mathcal{N}$ the intersection numbers $\langle \underline{X} \cdot \underline{\pi}^{\pm}_{(\mathcal{A},e)} \rangle \in K_0(k)_F \simeq F$ vanish. Moreover, since $\mathcal{N}$ is a $\otimes$-ideal, the intersection numbers $\langle \underline{X}^n \cdot \underline{\pi}^{\pm}_{(\mathcal{A},e)} \rangle$ vanish also for any integer $n \geq 1$, where $\underline{X}^n$ stands for the n-fold composition of $\underline{X}$. As proved in [30, Corollary 4.4], the intersection numbers $\langle \underline{X}^n \cdot \underline{\pi}^{\pm}_{(\mathcal{A},e)} \rangle$ agree with the categorical trace of the noncommutative correspondences $\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathcal{A},e)}$. Since (11.2) is symmetric monoidal, we conclude that the traces

$$\text{tr}(\overline{HP_*}(\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathcal{A},e)})) = \text{tr}(\overline{HP_*}(\underline{X})^n \circ \pi^{\pm}_{(\mathcal{A},e)}) = \text{tr}(\overline{HP_*}^{\pm}(\underline{X})^n), \quad n \geq 1,$$

vanish. Recall that over a field of characteristic zero, a nilpotent linear map can be characterized by the fact that the trace of all its n-fold compositions vanish. As a consequence, the F-linear transformations

$$\overline{HP_*}^{\pm}(\underline{X}) : \overline{HP_*}((\mathcal{A}, e)) \twoheadrightarrow \overline{HP_*}^{\pm}((\mathcal{A}, e)) \xrightarrow{\overline{HP_*}(\underline{X})} \overline{HP_*}^{\pm}((\mathcal{A}, e)) \hookrightarrow \overline{HP_*}((\mathcal{A}, e))$$

are nilpotent and so $\overline{HP_*}(\underline{X})$ is also nilpotent, Finally, since the functor (11.3) is faithful, the noncommutative correspondence $\underline{X}$ becomes nilpotent in $\text{NChow}(k)_F/\text{Ker}(\overline{HP_*})$. This completes the proof.

The case where k is a field extension of F is similar. The only difference is that (11.3) takes values in $\text{sVect}(k)$ instead of $\text{sVect}(F)$. Since by hypothesis k is a field of characteristic zero, the same reasoning applies. $\square$

Proposition 11.5. *Assume that F is a field extension of k or vice versa, and that the noncommutative standard conjecture $C_{NC}(\mathcal{A})$ holds for every smooth and proper dg category $\mathcal{A}$. Then the induced functor*

$$(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}))^\natural \to \mathrm{NNum}(k)_F \qquad (11.6)$$

is full, conservative, and (essentially) surjective.

Proof. Since we have an inclusion $\mathrm{Ker}(\overline{HP_*}) \subset \mathcal{N}$ of $\otimes$-ideals, the induced functor

$$\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}) \twoheadrightarrow \mathrm{NChow}(k)_F/\mathcal{N}$$

is clearly full. As explained by Bass [5, §III Prop. 2.10], idempotent elements can always be lifted along surjective homomorphisms with a nilpotent kernel. Hence, by Proposition 11.1, the functor (11.6) is also (essentially) surjective and conservative. $\qquad\square$

To simplify the proof of Theorem 1.4, we now introduce the following general result.

Proposition 11.7. *Let F be a field of characteristic zero and K a field extension of F, and consider two F-linear symmetric monoidal functors*

$$H : \mathcal{C} \to \mathrm{sVect}(K), \qquad P : \mathcal{C} \to \mathcal{D},$$

where $\mathrm{sVect}(K)$ denotes the category of finite-dimensional super K-vector spaces. Assume that H is faithful, P is (essentially) surjective, and for every object $N \in \mathcal{C}$, the Künneth projectors $\pi_N^\pm : H(N) \twoheadrightarrow H^\pm(N) \hookrightarrow H(N)$ can be written as $\pi_N^\pm = H(\underline{\pi}_N^\pm)$ with $\underline{\pi}_N^\pm \in \mathrm{End}_{\mathcal{C}}(N)$. Then, by modifying the symmetry isomorphism constraints of $\mathcal{C}$ and $\mathcal{D}$, we obtain new symmetric monoidal categories $\mathcal{C}^\dagger$ and $\mathcal{D}^\dagger$ and (composed) F-linear symmetric monoidal functors

$$\mathcal{C}^\dagger \xrightarrow{H} \mathrm{sVect}(K) \to \mathrm{Vect}(K), \qquad P : \mathcal{C}^\dagger \to \mathcal{D}^\dagger,$$

where $\mathrm{sVect}(K) \to \mathrm{Vect}(K)$ is the forgetful functor.

Proof. By applying [3, Prop. 8.3.1] to the functor H we obtain the symmetric monoidal category $\mathcal{C}^\dagger$ and the F-linear symmetric monoidal (composed) functor $\mathcal{C}^\dagger \xrightarrow{H} \mathrm{sVect}(K) \to \mathrm{Vect}(K)$. As explained in *loc. cit.*, the new symmetry isomorphism constraints are

$$c_{N_1,N_2}^\dagger := c_{N_1,N_2} \circ (e_{N_1} \otimes e_{N_2}), \qquad (11.8)$$

where e_N is the endomorphism $2 \cdot \underline{\pi}_N^+ - \mathrm{id}_N$ of N. Since by hypothesis the functor P is (essentially) surjective, we can use the image of e_N to modify the symmetry isomorphism constraints of $\mathcal{D}$ as in (11.8). In this way we obtain the symmetric monoidal category $\mathcal{D}^\dagger$ and the F-linear symmetric monoidal functor $P : \mathcal{C}^\dagger \to \mathcal{D}^\dagger$. $\qquad\square$

Proof of Theorem 1.4. Note first that since by hypothesis k is of characteristic zero, the (nontrivial) field F is also of characteristic zero. As explained in (the proofs of) Propositions 11.1 and 11.5, we have F-linear symmetric monoidal functors

$$\overline{HP_*} : (\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}))^\natural \to \mathrm{sVect}(K), \qquad (11.9)$$

$$(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}))^\natural \to \mathrm{NNum}(k)_F, \qquad (11.10)$$

with (11.9) faithful and (11.10) (essentially) surjective. The field K is equal to F when F is a field extension of k, and equal to k when k is a field extension of F. In both cases, K is a field extension of F. By hypothesis the noncommutative standard conjecture $C_{NC}(\mathcal{A})$ holds for every smooth and proper dg category $\mathcal{A}$, and so we can apply the above general Proposition 11.7 to the functors (11.9) and (11.10). We then obtain the diagram

$$
\begin{array}{ccc}
(\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}))^{\natural,\dagger} \xrightarrow{\overline{HP_*}} \mathrm{sVect}(K) \longrightarrow \mathrm{Vect}(K) \\
\downarrow \\
\mathrm{NNum}^\dagger(k)_F
\end{array}
\tag{11.11}
$$

Now, recall that by construction the category $\mathrm{NNum}^\dagger(k)_F$ is F-linear, additive, rigid symmetric monoidal, and such that the endomorphisms of its $\otimes$-unit is the field F. By Theorem 5.6 it is moreover abelian (semisimple). Hence, to prove that it is Tannakian, we can make use of Deligne's intrinsic characterization (Theorem 7.1). Concretely, we need to show that the rank $\mathrm{rk}(N)$ of every noncommutative numerical motive $N \in \mathrm{NNum}^\dagger(k)_F$ is a nonnegative integer. Since the vertical functor of diagram (11.11) is (essentially) surjective and restricts to an isomorphism between the endomorphisms of the corresponding $\otimes$-units, we conclude that $\mathrm{rk}(N) = \mathrm{rk}(\widetilde{N})$ for any lift $\widetilde{N} \in (\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*}))^{\natural,\dagger}$ of N. Moreover, since the (composed) horizontal functor of diagram (11.11) is faithful, we have $\mathrm{rk}(\widetilde{N}) = \mathrm{rk}(\overline{HP_*}(\widetilde{N}))$. Finally, since in the symmetric monoidal category $\mathrm{Vect}(K)$ the rank $\mathrm{rk}(\overline{HP_*}(\widetilde{N}))$ can be written as $\dim(\overline{HP_*}^+(\widetilde{N})) + \dim(\overline{HP_*}^-(\widetilde{N}))$, we conclude that $\mathrm{rk}(N)$ is a nonnegative integer. $\qquad\square$

12. Noncommutative homological motives

In this section we start by recalling the standard conjecture D. Then we prove Theorems 1.5 and 1.6 and along the way introduce the category of *noncommutative homological motives*. In what follows, Z will be a smooth projective k-scheme over a field of characteristic zero.

Recall from [2, §3.2] the definition of the F-vector spaces $\mathcal{Z}^*_{\mathrm{hom}}(Z)_F$ and $\mathcal{Z}^*_{\mathrm{num}}(Z)_F$ of algebraic cycles (of arbitrary codimension), where the homological equivalence relation is taken with respect to de Rham cohomology.

Standard conjecture $D(Z)$ (see [2, §5.4.1.1]). *The following equality holds:*

$$
\mathcal{Z}^*_{\mathrm{hom}}(Z)_F = \mathcal{Z}^*_{\mathrm{num}}(Z)_F.
$$

Now, recall from Theorem 9.2 that periodic cyclic homology gives rise to a well-defined F-linear symmetric monoidal functor

$$
\overline{HP_*} : \mathrm{NChow}(k)_F \to \mathrm{sVect}(K).
\tag{12.1}
$$

The field K is equal to F when F is a field extension of k, and equal to k when k is a field extension of F. Given a smooth and proper dg category $\mathcal{A}$, we then have an induced F-linear homomorphism

$$
K_0(\mathcal{A})_F = \mathrm{Hom}_{\mathrm{NChow}(k)_F}(\underline{k}, \mathcal{A}) \xrightarrow{\overline{HP_*}} \mathrm{Hom}_{\mathrm{sVect}(K)}(\overline{HP_*}(\underline{k}), \overline{HP_*}(\mathcal{A})).
$$

The associated kernel gives rise to a well-defined equivalence relation on $K_0(\mathcal{A})_F$ that we denote by $\sim_{\text{hom}}$. On the other hand, as explained in §3.2, the noncommutative correspondences which are numerically equivalent to zero form an F-linear subspace of $K_0(\mathcal{A})_F = \text{Hom}_{\text{NChow}(k)_F}(k, \mathcal{A})$ and hence give rise to an equivalence relation on $K_0(\mathcal{A})_F$ that we denote by $\sim_{\text{num}}$.

Proof of Theorem 1.5. As explained in the proof of Theorem 1.3, the composition

$$\text{Chow}(k)_F \xrightarrow{\tau} \text{Chow}(k)_F/_{-\otimes\mathbb{Q}(1)} \xrightarrow{R} \text{NChow}(k)_F \xrightarrow{\overline{HP_*}} \text{sVect}(k) \qquad (12.2)$$

agrees with the functor $s\overline{H^*_{dR}}$ (see (10.8)) associated to de Rham cohomology. Hence, its kernel *Ker* agrees with the kernel of the symmetric monoidal functor

$$\overline{H^*_{dR}} : \text{Chow}(k)_F \to \text{GrVect}(k).$$

From (5.5) we then obtain the commutative diagram

$$
\begin{array}{ccccc}
\text{Chow}(k)_F/\mathit{Ker} & \xrightarrow{\Psi} & (\text{Chow}(k)_F/_{-\otimes\mathbb{Q}(1)})/\text{Ker}(\overline{HP_*} \circ R) & \xrightarrow{\Phi} & \text{NChow}(k)_F/\text{Ker}(\overline{HP_*}) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Num}(k)_F & \xrightarrow{\tau} & \text{Num}(k)_F/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{R_\mathcal{N}} & \text{NNum}(k)_F
\end{array}
$$

where *Ker* denotes the kernel of the composed functor (12.2). Now, by Lemma 4.7 (with $\mathcal{C} = \text{Chow}(k)_F$, $\mathcal{O} = \mathbb{Q}(1)$, and $H = \overline{HP_*} \circ R$), the functor Ψ admits the factorization

$$\text{Chow}(k)_F/\mathit{Ker} \xrightarrow{\tau} (\text{Chow}(k)_F/\mathit{Ker})/_{-\otimes\mathbb{Q}(1)} \xrightarrow{\Omega} (\text{Chow}(k)_F/_{-\otimes\mathbb{Q}(1)})/\text{Ker}(\overline{HP_*} \circ R).$$

As explained in the proof of Theorem 1.3, the image of Z under the composed functor $\Phi \circ \Psi$ is naturally isomorphic to the dg category $\mathcal{D}^{\text{dg}}_{\text{perf}}(Z)$. Similarly, the image of the affine k-scheme $\text{spec}(k)$ is naturally isomorphic to $\mathcal{D}^{\text{dg}}_{\text{perf}}(\text{spec}(k)) \simeq k$. We can then consider the induced commutative square

$$
\begin{array}{ccc}
(\text{Chow}(k)_F/\mathit{Ker})/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{\Phi\circ\Omega} & \text{NChow}(k)_F/\text{Ker}(\overline{HP_*}) \\
\downarrow & & \downarrow \\
\text{Num}(k)_F/_{-\otimes\mathbb{Q}(1)} & \xrightarrow{R_\mathcal{N}} & \text{NNum}(k)_F
\end{array}
$$

and the associated commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{(\text{Chow}(k)_F/\text{Ker})/_{-\otimes\mathbb{Q}(1)}}(\text{spec}(k), Z) & \longrightarrow & \text{Hom}_{\text{NChow}(k)_F/\text{Ker}(\overline{HP_*})}(k, \mathcal{D}^{\text{dg}}_{\text{perf}}(Z)) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Num}(k)_F/_{-\otimes\mathbb{Q}(1)}}(\text{spec}(k), Z) & \longrightarrow & \text{Hom}_{\text{NNum}(k)_F}(k, \mathcal{D}^{\text{dg}}_{\text{perf}}(Z))
\end{array}
$$

By combining the above arguments with the constructions of the categories $\mathrm{Chow}(k)_F$, $\mathrm{Num}(k)_F$, $\mathrm{NChow}(k)_F$ and $\mathrm{NNum}(k)_F$, we observe that the preceding commutative square identifies with

$$
\begin{array}{ccc}
\mathcal{Z}^*_{\mathrm{hom}}(Z)_F & \longrightarrow\!\!\!\!\!\rightarrow & K_0(\mathcal{D}^{\mathrm{dg}}_{\mathrm{perf}}(Z))_F/\!\!\sim_{\mathrm{hom}} \\
\downarrow & & \downarrow \\
\mathcal{Z}^*_{\mathrm{num}}(Z)_F & \xrightarrow{\ \sim\ } & K_0(\mathcal{D}^{\mathrm{dg}}_{\mathrm{perf}}(Z))_F/\!\!\sim_{\mathrm{num}}
\end{array}
\tag{12.3}
$$

Since the functor $R_{\mathcal{N}}$ is fully-faithful, the lower horizontal map is an isomorphism. Recall from Lemma 4.7 that the functor Ω is full. Hence, since the induced functor Φ is fully-faithful, the upper horizontal map in (12.3) is surjective. Now, suppose that the standard conjecture $D(Z)$ holds, i.e. the left vertical map of diagram (12.3) is an isomorphism. The commutativity of the diagram combined with all the above facts implies that the right vertical map in (12.3) is injective. Since by construction it is already surjective, we conclude that it is an isomorphism. This is precisely the statement of the noncommutative standard conjecture $D_{NC}(\mathcal{D}^{\mathrm{dg}}_{\mathrm{perf}}(Z))$. $\qquad\square$

In analogy with the commutative world we introduce the category of noncommutative homological motives.

Definition 12.4. Let k and F be fields with F a field extension of k or vice versa. The category $\mathrm{NHom}(k)_F$ of *noncommutative homological motives* is the pseudo-abelian envelope of $\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})$, where $\mathrm{Ker}(\overline{HP_*})$ is the kernel of (12.1).

As mentioned in §3.2, $\mathcal{N}$ is the largest $\otimes$-ideal of $\mathrm{NChow}(k)_F$ (distinct from the entire category). Hence, the inclusion of $\otimes$-ideals $\mathrm{Ker}(\overline{HP_*}) \subset \mathcal{N}$ gives rise to an induced functor

$$
\mathrm{NHom}(k)_F \to \mathrm{NNum}(k)_F.
\tag{12.5}
$$

Proposition 12.6. *Let k be a field of characteristic zero. If the noncommutative standard conjectures $C_{NC}(\mathcal{A})$ and $D_{NC}(\mathcal{A})$ hold for every smooth and proper dg category $\mathcal{A}$, then the induced functor (12.5) is an equivalence of categories.*

Proof. By Proposition 11.5 the functor (12.5) is full and (essentially) surjective. It remains to show that it is faithful. Given noncommutative Chow motives $(\mathcal{A}, e)$ and $(\mathcal{B}, e')$ (see §3.1) we need to show that the induced F-linear homomorphism

$$
\mathrm{Hom}_{\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})}((\mathcal{A}, e), (\mathcal{B}, e')) \to \mathrm{Hom}_{\mathrm{NNum}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e'))
$$

is faithful. For this, it suffices to show that the induced F-linear homomorphism

$$
\mathrm{Hom}_{\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{\mathrm{NNum}(k)_F}(\mathcal{A}, \mathcal{B})
\tag{12.7}
$$

is faithful. The categorical dual of $\mathcal{A}$ is its opposite dg category $\mathcal{A}^{\mathrm{op}}$, and so by adjunction, (12.7) identifies with the homomorphism

$$
\mathrm{Hom}_{\mathrm{NChow}(k)_F/\mathrm{Ker}(\overline{HP_*})}(\underline{k}, \mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B}) \to \mathrm{Hom}_{\mathrm{NNum}(k)_F}(\underline{k}, \mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B}).
$$

By the definition of $\mathrm{NChow}(k)_F$ and $\mathrm{NNum}(k)_F$, the above homomorphism identifies with the canonical homomorphism

$$K_0(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B})_F/\sim_{\mathrm{hom}} \to K_0(\mathcal{A}^{\mathrm{op}} \otimes_k \mathcal{B})_F/\sim_{\mathrm{num}}. \tag{12.8}$$

Finally, since by hypothesis the noncommutative standard conjecture $D_{NC}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ holds, i.e. (12.8) is an isomorphism, the homomorphism (12.7) is faithful. $\square$

Proof of Theorem 1.6. Recall from the proof of Theorem 1.4 the construction of the following diagram

$$\begin{array}{ccc}
\mathrm{NHom}^\dagger(k)_F & \xrightarrow{\overline{HP_*}} & \mathrm{sVect}(F) \longrightarrow \mathrm{Vect}(F) \\
\downarrow & & \\
\mathrm{NNum}^\dagger(k)_F & &
\end{array}$$

Proposition 12.6 implies that the vertical functor in the above diagram is an equivalence of categories. As a consequence, we obtain an exact faithful $\otimes$-functor

$$\overline{HP_*} : \mathrm{NNum}^\dagger(k)_F \to \mathrm{Vect}(F). \tag{12.9}$$

Recall from Theorem 1.4 that the category $\mathrm{NNum}^\dagger(k)_F$ is Tannakian. The functor (12.9) shows that $\mathrm{NNum}^\dagger(k)_F$ is moreover neutral and so the proof is finished. $\square$

13. Tate triples and quotient categories

In this section we recall the notion of Tate triples from Deligne–Milne [15, §5] and the notion of a quotient category from Milne [32]. In what follows, F will be a field of characteristic zero and $(\mathcal{C}, \otimes, \mathbf{1})$ an F-linear, abelian, rigid symmetric monoidal category such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq F$.

Let μ_n be the affine group scheme of the nth *roots of unity*, i.e. the kernel of the nth-power morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$. In particular, μ_2 is the affine group scheme dual to the Hopf algebra $F[t]/(t^2 - 1)$.

Definition 13.1. Let $A = \mathbb{Z}$ (resp. $\mathbb{Z}/2$) and let B be the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$ (resp. μ_2). An A-*grading* on a Tannakian category $\mathcal{C}$ can be variously described as:

(i) a functorial A-grading on objects $X = \bigoplus X^a$ compatible with tensor products $(X \otimes Y)^a = \bigoplus_{a=b+c} X^b \otimes Y^c$;

(ii) an A-grading of the identity functor $\mathrm{id}_{\mathcal{C}}$ compatible with tensor products;

(iii) a morphism $B \to \underline{\mathrm{Aut}}^\otimes(\mathrm{id}_{\mathcal{C}})$;

(iv) a central morphism $B \to \underline{\mathrm{Aut}}^\otimes(\omega)$ for every fiber functor ω.

Definition 13.2. A *Tate triple* $\mathcal{T} = (\mathcal{C}, w, T)$ consists of a Tannakian category $\mathcal{C}$, a $\mathbb{Z}$-grading $w : \mathbb{G}_{\mathrm{m}} \to \underline{\mathrm{Aut}}^\otimes(\mathrm{id}_{\mathcal{C}})$ (called the *weight grading*), and an invertible object T (called the *Tate object*) of weight -2. Given an object $X \in \mathcal{C}$ and an integer n, we will write $X(n)$ for $X \otimes T^{\otimes n}$. A K-*valued fiber functor on* $\mathcal{T}$ (with K a field extension of F) is a K-valued fiber functor ω on $\mathcal{C}$ endowed with an isomorphism $\omega(T) \simeq \omega(T(1))$. If $\mathcal{T}$ admits an F-valued fiber functor, then $\mathcal{T}$ is called a *neutral Tate triple*.

Example 13.3. (i) Let G be an affine group scheme endowed with a central morphism $w : \mathbb{G}_m \to G$ and with a morphism $t : G \to \mathbb{G}_m$ such that $t \circ w = -2$. Let T be the representation of G on F such that $g \in G$ acts as multiplication by $t(g)$. Then, as explained in [15, Example 5.4], $\mathcal{T} = (\mathrm{Rep}_F(G), w, T)$ is a neutral Tate triple.

(ii) Let k be a field of characteristic zero. If we assume that the standard conjecture $C(Z)$ holds for every smooth projective k-scheme Z (see §10), then $\mathcal{T} = (\mathrm{Num}^\dagger(k)_\mathbb{Q}, w, \mathbb{Q}(1))$ is a Tate triple (see [15, Thm. 6.7]). The $\mathbb{Z}$-grading w is given by $(Z, p, m)_i := (Z, (p \circ \underline{\pi}_Z^{2m+i})(Z), m)$, where $\underline{\pi}_Z^n$ are algebraic correspondences such that $\overline{H_{dR}^*}(\underline{\pi}_Z^n) = \pi_Z^n$. The Tannakian category $\mathrm{Num}^\dagger(k)_\mathbb{Q}$ is obtained from $\mathrm{Num}(k)_\mathbb{Q}$ by modifying its symmetry isomorphism constraints using the algebraic correspondences $\underline{\pi}_Z^n$. Moreover, if we assume that the standard conjecture $D(Z)$ holds for every smooth projective k-scheme Z (see §12), then de Rham cohomology H_{dR}^* give rise to a k-valued fiber functor on $\mathcal{T}$.

As explained in [15, Prop. 5.5], every Tate triple $\mathcal{T} = (\mathcal{C}, w, T)$ gives use to a central morphism $w : \mathbb{G}_m \to \mathrm{Gal}(\mathcal{C})$ and to a morphism $t : \mathrm{Gal}(\mathcal{C}) \to \mathbb{G}_m$ such that $t \circ w = -2$.

Lemma 13.4 (see [15, §5]). *Let G be an affine group scheme endowed with a central morphism $w : \mathbb{G}_m \to G$ and with a homomorphism $t : G \to \mathbb{G}_m$ such that $t \circ w = -2$. Consider the kernel $G_0 := \mathrm{Ker}(t : G \to \mathbb{G}_m)$ and the associated Tannakian category $\mathrm{Rep}_F(G_0)$. Then:*

(i) *The $\mathbb{Z}$-grading on $\mathrm{Rep}_F(G)$ induces a $\mathbb{Z}/2$-grading $\epsilon : \mu_2 \to G_0$ on $\mathrm{Rep}_F(G_0)$, making it into a $\mathbb{Z}/2$-graded Tannakian category.*

(ii) *The inclusion $G_0 \hookrightarrow G$ gives rise to an (essentially) surjective $\otimes$-functor $Q : \mathrm{Rep}_F(G) \to \mathrm{Rep}_F(G_0)$, which maps homogeneous objects of weight n to homogeneous objects of weight n (mod 2) and the Tate object T to the $\otimes$-unit $\mathbf{1}$.*

(iii) *The Tate object T becomes an identity object in $\mathrm{Rep}_F(G_0)$, i.e. $T \simeq T \otimes T$. Moreover, the functor $\tau : \mathrm{Rep}_F(G_0) \to \mathrm{Rep}_F(G_0)$ given by $X \mapsto X \otimes T$ is an equivalence of categories.*

(iv) *Two homogeneous objects $X, Y \in \mathrm{Rep}_F(G)$ of respective weights n and m become isomorphic in $\mathrm{Rep}_F(G_0)$ if and only if $m - n = 2\ell$ for some $\ell \in \mathbb{Z}$ and $X(\ell) \simeq Y$.*

Quotient categories

Let F be an arbitrary field.

Theorem 13.5 (see Milne [33]). *Let $S \subset \mathcal{C}$ be a fully-faithful inclusion of neutral Tannakian categories with associated affine group schemes $\mathrm{Gal}(S)$ and $\mathrm{Gal}(\mathcal{C})$. Then:*

(i) *There is a quotient neutral Tannakian category $\mathcal{Q}$ and an exact $\otimes$-functor $Q : \mathcal{C} \to \mathcal{Q}$ such that all objects of $\mathcal{Q}$ are subquotients of objects in the image of Q. Moreover, the objects of S are precisely those objects of $\mathcal{C}$ which become trivial in $\mathcal{Q}$.*

(ii) *The fully-faithful inclusion $S \subset \mathcal{C}$ gives rise to a faithfully flat morphism $\mathrm{Gal}(\mathcal{C}) \twoheadrightarrow \mathrm{Gal}(S)$ of affine group schemes.*

(iii) *Let H be the kernel of the morphism* $\mathrm{Gal}(\mathcal{C}) \twoheadrightarrow \mathrm{Gal}(\mathcal{S})$. *Then there is an equivalence of categories* $\mathrm{Rep}_F(H) \simeq \mathcal{Q}$.

(iv) *Given an object* $X \in \mathcal{C}$, *one denotes by* X^H *the largest subobject of X on which H acts trivially. Under this notation, the subcategory* $\mathcal{S} \subset \mathcal{C}$ *coincides with the Tannakian subcategory* $\mathcal{C}^H \subset \mathcal{C}$ *of objects X such that* $X^H = X$.

Definition 13.6. Let $\mathcal{C}$ be a neutral Tannakian category and $\omega_0 : \mathcal{S} \to \mathrm{Vect}(F)$ an F-valued fiber functor on a Tannakian subcategory $\mathcal{S} \subset \mathcal{T}$. Then the *quotient category* $\mathcal{C}/\omega_0$ is the pseudo-abelian envelope of the category $\mathcal{C}'$ which has the same objects as $\mathcal{C}$ and morphisms

$$\mathrm{Hom}_{\mathcal{C}'}(X, Y) := \omega_0(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^H).$$

Here, $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ stands for the internal Hom-object of the symmetric monoidal category $\mathcal{C}$, and $H \subset \mathrm{Gal}(\mathcal{C})$ for the group subscheme described in Theorem 13.5(iii).

14. Motivic Galois groups

In this section we prove Theorem 1.7.

Proposition 14.1. *Let F be a field of characteristic zero and* $\mathcal{T} = (\mathcal{C}, w, T)$ *a neutral Tate triple. Denote by $\mathcal{S}$ the full neutral Tannakian subcategory of $\mathcal{C}$ generated by the Tate object T. Then the pseudo-abelian envelope of the orbit category* $\mathcal{C}/_{-\otimes T}$ *(see §4) is a neutral Tannakian category and the sequence of exact $\otimes$-functors* $\mathcal{S} \subset \mathcal{C} \to (\mathcal{C}/_{-\otimes T})^\natural$ *induces a group scheme isomorphism*

$$\mathrm{Gal}((\mathcal{C}/_{-\otimes T})^\natural) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{Gal}(\mathcal{C}) \twoheadrightarrow \mathbb{G}_m).$$

Proof. By Theorem 13.5(ii) the fully-faithful inclusion $\mathcal{S} \subset \mathcal{C}$ gives rise to a faithfully flat morphism $t : \mathrm{Gal}(\mathcal{C}) \twoheadrightarrow \mathrm{Gal}(\mathcal{S})$ of affine group schemes. Let us denote by H its kernel. Thanks to items (i) and (iii) of Theorem 13.5, we have a sequence of exact $\otimes$-functors $\mathcal{S} \subset \mathcal{C} \to \mathrm{Rep}_F(H)$ inducing a group scheme isomorphism

$$\mathrm{Gal}(\mathrm{Rep}_F(H)) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{Gal}(\mathcal{C}) \twoheadrightarrow \mathrm{Gal}(\mathcal{S})).$$

The proof will consist in showing that the categories $\mathrm{Rep}_F(H)$ and $(\mathcal{C}/_{-\otimes T})^\natural$ are $\otimes$-equivalent and that the affine group scheme $\mathrm{Gal}(\mathcal{S})$ is isomorphic to $\mathbb{G}_m$.

Let us start with the latter claim. The $\mathbb{Z}$-grading w of the Tate triple structure implies that the fiber functor $\omega : \mathcal{C} \to \mathrm{Vect}(F)$ factors as follows:

where $\mathrm{GrVect}(F)$ denotes the category of finite-dimensional $\mathbb{Z}$-graded F-vector spaces and U the forgetful functor $\{V_n\}_{n \in \mathbb{Z}} \mapsto \bigoplus_{n \in \mathbb{Z}} V_n$. Since the Tate object T is invertible and the functor $\overline{\omega}$ is symmetric monoidal, the object $\overline{\omega}(T)$ is also invertible. Moreover, since T is of degree two and the invertible objects in $\mathrm{GrVect}(F)$ are one-dimensional, the $\mathbb{Z}$-graded F-vector space $\overline{\omega}(T)$ is also of degree two and one-dimensional. In particular,

$\text{End}_{\text{GrVect}(F)}(\overline{\omega}(T)) \simeq F$. This implies that $\overline{\omega}$ establishes a $\otimes$-equivalence between $\mathcal{S}$ and the Tannakian category generated by $\overline{\omega}(T)$. The latter category identifies with the subcategory $\text{GrVect}^+(F) \subset \text{GrVect}(F)$ of evenly supported $\mathbb{Z}$-graded F-vector spaces. Via renumbering the indices, this category is $\otimes$-equivalent to $\text{GrVect}(F)$. As a consequence, the affine group scheme $\text{Gal}(\mathcal{S})$ (with respect to ω) agrees with the affine group scheme of the Tannakian category $\text{GrVect}(F)$ (with respect to U). It is well-known that the latter affine group scheme is isomorphic to $\mathbb{G}_m$ and so the proof of the above claim is finished.

Let us now show that the categories $\text{Rep}_F(H)$ and $(\mathcal{C}/_{-\otimes T})^\natural$ are $\otimes$-equivalent. Since by hypothesis $\mathcal{T} = (\mathcal{C}, w, T)$ is a Tate triple, Lemma 13.4(ii) (with $G = \text{Gal}(\mathcal{C})$ and $G_0 = H$) implies that the quotient functor $Q : \mathcal{C} \to \text{Rep}_F(H)$ maps the Tate object T to the $\otimes$-unit of $\text{Rep}_F(H)$. As a consequence, we have a natural 2-isomorphism $Q \circ (-\otimes T) \xrightarrow{\sim} Q$. By the 2-universality property of the orbit category $\mathcal{C}/_{-\otimes T}$ and the fact that the category $\text{Rep}_F(H)$ is idempotent complete, we obtain an induced symmetric monoidal functor

$$(\mathcal{C}/_{-\otimes T})^\natural \to \text{Rep}_F(H). \tag{14.2}$$

Now, recall from [32, Example 2.6] that since $(\mathcal{C}, w, T)$ is a Tate triple, the category $\text{Rep}_F(H)$ can be identified with the quotient category $\mathcal{C}/\omega_0$ with respect to the F-valued fiber functor

$$\omega_0 : \mathcal{S} \to \text{Vect}(F), \quad X \mapsto \text{colim}_n \text{Hom}_{\mathcal{C}}\Big(\bigoplus_{r=-n}^{n} \mathbf{1}(r), X\Big).$$

As explained in Definition 13.6, $\mathcal{C}/\omega_0$ is the pseudo-abelian envelope of a certain category $\mathcal{C}'$. Hence, it suffices to show that $\mathcal{C}/_{-\otimes T}$ and $\mathcal{C}'$ are $\otimes$-equivalent. By construction, they have the same objects. As for their morphisms, by arguing as in [32, Prop. 2.3], we conclude that

$$\text{Hom}_{\mathcal{C}'}(X, Y) = \text{colim}_n \bigoplus_{r=-n}^{n} \text{Hom}_{\mathcal{C}}(\mathbf{1}, \underline{\text{Hom}}(X \otimes T^{\otimes r}, Y))$$

$$= \text{colim}_n \bigoplus_{r=-n}^{n} \text{Hom}_{\mathcal{C}}(X \otimes T^{\otimes r}, Y) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(X, Y \otimes T^{\otimes j}).$$

The latter description agrees with the one of the orbit category $\mathcal{C}/_{-\otimes T}$ (see (4.1) with $\mathcal{O} = T$). As a consequence, (14.2) is a $\otimes$-equivalence. $\square$

The notion of Tate triple does not admit an immediate generalization to the super-Tannakian setting. Motivated by the standard theory of motives, we nevertheless introduce the following notion.

Definition 14.3. A *super-Tate triple* $\mathcal{ST} = (\mathcal{C}, \omega, \underline{\pi}_X^\pm, \mathcal{T}^\dagger)$ consists of:

(i) a neutral super-Tannakian category $\mathcal{C}$;
(ii) a super-fiber functor $\omega : \mathcal{C} \to \text{sVect}(F)$;
(iii) idempotent endomorphisms $\underline{\pi}_X^\pm \in \text{End}_{\mathcal{C}}(X)$, $X \in \mathcal{C}$, such that $\omega(\underline{\pi}_X^\pm) = \pi_X^\pm$, where $\pi_X^\pm$ are the Künneth projectors;
(iv) a neutral Tate triple $\mathcal{T}^\dagger = (\mathcal{C}^\dagger, w, T)$ on the category $\mathcal{C}^\dagger$. Recall from Proposition 11.7 that $\mathcal{C}^\dagger$ is obtained from $\mathcal{C}$ by modifying the symmetry isomorphism constraints through the use of the endomorphisms $\underline{\pi}_X^\pm$.

Example 14.4. Let k be a field of characteristic zero. If we assume that the standard conjectures $C(Z)$, $D(Z)$ hold for every smooth projective k-scheme Z, then the data

$$(\mathrm{Num}(k)_k, s\overline{H_{dR}^*}, \underline{\pi}_X^{\pm}, (\mathrm{Num}^\dagger(k)_k, w, \mathbb{Q}(1)))$$

is a super-Tate triple. The super-fiber functor $s\overline{H_{dR}^*} : \mathrm{Num}(k)_k \to \mathrm{sVect}(k)$ is the one associated to de Rham cohomology and $(\mathrm{Num}^\dagger(k)_k, w, \mathbb{Q}(1))$ is the neutral Tate triple obtained from the one of Example 13.3(ii) by extension of scalars along the functor $- \otimes_{\mathbb{Q}} k$ (see [32, §1]).

Proposition 14.5. *Let F be a field of characteristic zero and $\mathcal{ST} = (\mathcal{C}, \omega, \underline{\pi}_X^{\pm}, \mathcal{T}^\dagger)$ a super-Tate triple. Let $\mathcal{S}$ denote the full neutral super-Tannakian subcategory of $\mathcal{C}$ generated by the Tate object T. Assume the following conditions hold:*

(i) *The Tate object T is such that $\underline{\pi}_T^-(T) = 0$.*

(ii) *Let $\epsilon : \mu_2 \to H$ be the $\mathbb{Z}/2$-grading induced by the $\mathbb{Z}$-grading on the neutral Tate triple $\mathcal{T}^\dagger$ as in Lemma 13.4 (with $G = \mathrm{Gal}(\mathcal{C}^\dagger)$ and $G_0 = H$). Then the affine super-group scheme (H, ϵ) is isomorphic to the kernel of the induced faithfully flat morphism $\mathrm{sGal}(\mathcal{C}) \twoheadrightarrow \mathrm{sGal}(\mathcal{S})$.*

(iii) *The super-Tannakian category $\mathrm{Rep}_F((H, \epsilon))$ of finite-dimensional F-valued super representations is such that $\mathrm{Rep}_F^\dagger((H, \epsilon)) \simeq Q$, where $Q \simeq \mathrm{Rep}_F(H)$ is the quotient Tannakian category associated to the fully-faithful inclusion $\mathcal{S}^\dagger \subset \mathcal{C}^\dagger$ (see Theorem 13.5(ii)–(iii)).*

Then the pseudo-abelian envelope of the orbit category $\mathcal{C}/_{-\otimes T}$ is a neutral super-Tannakian category and the sequence of exact $\otimes$-functors $\mathcal{S} \subset \mathcal{C} \to (\mathcal{C}/_{-\otimes T})^\natural$ induces a super-group scheme isomorphism

$$\mathrm{sGal}((\mathcal{C}/_{-\otimes T})^\natural) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{sGal}(\mathcal{C}) \twoheadrightarrow \mathbb{G}_m).$$

Proof. By condition (ii), the sequence of exact of $\otimes$-functors $\mathcal{S} \subset \mathcal{C} \to \mathrm{Rep}_F((H, \epsilon))$ induces a super-group scheme isomorphism

$$\mathrm{sGal}(\mathrm{Rep}_F((H, \epsilon))) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{sGal}(\mathcal{C}) \twoheadrightarrow \mathrm{sGal}(\mathcal{S})).$$

Hence, the proof will consist in showing that the categories $\mathrm{Rep}_F((H, \epsilon))$ and $\mathcal{C}/_{-\otimes T}$ are $\otimes$-equivalent and that the affine super-group scheme $\mathrm{sGal}(\mathcal{S})$ is isomorphic to $\mathbb{G}_m$.

Let us start with the latter claim. Recall from the proof of Proposition 11.7 that the category $\mathcal{S}^\dagger$ is obtained from $\mathcal{S}$ by modifying its symmetry isomorphism constraints c_{N_1, N_2}. The new constraints are given by

$$c_{N_1, N_2}^\dagger := c_{N_1, N_2} \circ (e_{N_1} \otimes e_{N_2})$$

where $e_N = 2 \cdot \underline{\pi}_N^+ - \mathrm{id}_N$. Since by hypothesis $\underline{\pi}_T^-(T) = 0$ we conclude that $\underline{\pi}_N^-(N) = 0$ for every object $N \in \mathcal{S}$. As a consequence, $\underline{\pi}_N^+ = \mathrm{id}_N$ and so $e_N = \mathrm{id}_N$. The symmetric monoidal category $\mathcal{S}^\dagger$ is thus equal to $\mathcal{S}$. This implies that $\mathrm{sGal}(\mathcal{S}) \simeq \mathrm{Gal}(\mathcal{S}^\dagger)$ since the super-fiber functor $\omega : \mathcal{S} \to \mathrm{sVect}(F)$ takes values in the subcategory $\mathrm{sVect}^+(F) \subset$

$\mathrm{sVect}(F)$ of evenly supported super F-vector spaces, which via the forgetful functor is $\otimes$-equivalent to $\mathrm{Vect}(F)$. As explained in the proof of Proposition 14.1, $\mathrm{Gal}(\mathcal{S}^\dagger) \simeq \mathbb{G}_{\mathrm{m}}$ and so $\mathrm{sGal}(\mathcal{S}) \simeq \mathbb{G}_{\mathrm{m}}$.

Let us now show that $\mathrm{Rep}_F((H, \epsilon))$ and $\mathcal{C}/_{-\otimes T}$ are $\otimes$-equivalent. As in the Tannakian case, we have a sequence of exact $\otimes$-functors $\mathcal{S} \subset \mathcal{C} \to \mathrm{Rep}_F((H, \epsilon))$ such that T is mapped to the $\otimes$-unit of $\mathrm{Rep}_F((H, \epsilon))$. By the 2-universality property of category $\mathcal{C}/_{-\otimes T}$ and the fact that $\mathrm{Rep}_F((H, \epsilon))$ is idempotent complete, we obtain an induced symmetric monoidal functor

$$(\mathcal{C}/_{-\otimes T})^\natural \to \mathrm{Rep}_F((H, \epsilon)). \tag{14.6}$$

By combining Lemma 4.9 (with $\mathcal{O} = T$) with condition (iii), we obtain an induced functor

$$(\mathcal{C}^\dagger/_{-\otimes T})^\natural \xrightarrow{\sim} (\mathcal{C}/_{-\otimes T})^{\natural,\dagger} \to \mathrm{Rep}_F^\dagger((H, \epsilon)) \xrightarrow{\sim} \mathrm{Rep}_F(H). \tag{14.7}$$

Now, since by hypothesis $\mathcal{T}^\dagger$ is a neutral Tate triple, the proof of Proposition 14.1 shows us that (14.7) is a $\otimes$-equivalence, hence so is (14.6). $\square$

Proof of Theorem 1.7. Let us start by constructing the faithfully flat morphism (1.8). Since k is a field of characteristic zero, Theorem 1.2 implies that $\mathrm{NNum}(k)_k$ is super-Tannakian. Moreover, since the noncommutative standard conjectures $C_{NC}(\mathcal{A})$ and $D_{NC}(\mathcal{A})$ hold for every smooth and proper dg category $\mathcal{A}$, Proposition 12.6 implies that periodic cyclic homology gives rise to a super-fiber functor

$$\overline{HP_*} : \mathrm{NNum}(k)_k \to \mathrm{sVect}(k). \tag{14.8}$$

Now, recall from (5.5) the composition

$$\mathrm{Num}(k)_k \xrightarrow{\tau} \mathrm{Num}(k)_k/_{-\otimes \mathbb{Q}(1)} \xrightarrow{R_\mathcal{N}} \mathrm{NNum}(k)_k. \tag{14.9}$$

By composing (14.8) with (14.9) we obtain a well-defined super-fiber functor on $\mathrm{Num}(k)_k$ which, as explained in the proof of Theorem 1.3, is given by

$$s\overline{H_{dR}^*} : \mathrm{Num}(k)_k \to \mathrm{sVect}(k), \quad Z \mapsto \Big(\bigoplus_{n \text{ even}} H_{dR}^n(Z), \bigoplus_{n \text{ odd}} H_{dR}^n(Z) \Big).$$

Now, recall from Example 14.4 that since by hypothesis the standard conjectures $C(Z)$ and $D(Z)$ hold for every smooth projective k-scheme Z, we have a super-Tate triple

$$(\mathrm{Num}(k)_k, s\overline{H_{dR}^*}, \underline{\pi}_X^\pm, (\mathrm{Num}^\dagger(k)_k, w, \mathbb{Q}(1))).$$

Let us now show that this triple satisfies conditions (i)–(iii) of Proposition 14.5. As for (i), the $\mathbb{Z}$-graded k-vector space $\overline{H_{dR}^*}(\mathbb{Q}(1))$ is of degree two and one-dimensional. This implies that $\underline{\pi}_{\mathbb{Q}(1)}^+ = \mathrm{id}_{\mathbb{Q}(1)}$ and so $\underline{\pi}_{\mathbb{Q}(1)}^-(\mathbb{Q}(1)) = 0$. Item (ii) follows from [14, Example 0.4] and [15, Example 5.4] since the Tate triple $(\mathrm{Num}^\dagger(k)_k, w, \mathbb{Q}(1))$ can be written as $(\mathrm{Rep}(G), w, \mathbb{Q}(1))$ with $G = \mathrm{Gal}(\mathrm{Num}^\dagger(k)_k)$. For (iii), recall from Example 13.3(ii) that the $\mathbb{Z}$-grading w on $\mathrm{Num}^\dagger(k)_k$ is induced by the algebraic correspondences $\underline{\pi}_Z^n$ such that $\overline{H_{dR}^*}(\underline{\pi}_Z^n) = \pi_Z^n$. This implies that the $\mathbb{Z}/2$-grading ϵ agrees with the

$\mathbb{Z}/2$-grading induced by the algebraic correspondences $\underline{\pi}_Z^{\pm}$ such that $s\overline{H_{dR}^*}(\underline{\pi}_Z^{\pm}) = \pi_Z^{\pm}$. As a consequence, the category $\mathrm{Rep}((H, \epsilon))$ is obtained from $\mathrm{Rep}(H)$ by modifying its symmetry isomorphism constraints with respect to the $\mathbb{Z}/2$-grading induced by the algebraic correspondences $\underline{\pi}_Z^{\pm}$. Now, by Proposition 14.5 we obtain an induced super-group scheme isomorphism

$$\mathrm{sGal}((\mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)})^{\natural}) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{sGal}(\mathrm{Num}(k)_k) \twoheadrightarrow \mathbb{G}_m). \quad (14.10)$$

On the other hand, since $\mathrm{NNum}(k)_k$ is idempotent complete and semisimple (see Theorem 5.6), the functor $R_{\mathcal{N}}$ in diagram (14.9) gives rise to a faithfully flat morphism

$$\mathrm{sGal}(\mathrm{NNum}(k)_k) \twoheadrightarrow \mathrm{sGal}((\mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)})^{\natural}).$$

By combining it with (14.10) we finally obtain the morphism (1.8) of Theorem 1.7.

Let us now construct the faithfully flat morphism (1.9). We start by showing that through the modification of the symmetry isomorphism constraints of $\mathrm{NNum}(k)_k$ (as in Theorem 1.4), the sequence of functors

$$\mathrm{Num}(k)_k \xrightarrow{\tau} \mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)} \xrightarrow{R_{\mathcal{N}}} \mathrm{NNum}(k)_k$$

described in diagram (5.5) gives rise to a sequence

$$\mathrm{Num}^{\dagger}(k)_k \xrightarrow{\tau} \mathrm{Num}^{\dagger}(k)_k/_{-\otimes\mathbb{Q}(1)} \xrightarrow{R_{\mathcal{N}}} \mathrm{NNum}^{\dagger}(k)_k. \quad (14.11)$$

In order to show this, consider the diagram

$$
\begin{array}{ccccccc}
\mathrm{Chow}(k)_k & \longrightarrow & \mathrm{Chow}(k)_k/_{-\otimes\mathbb{Q}(1)} & \longrightarrow & \mathrm{NChow}(k)_k & \xrightarrow{\overline{HP_*}} & \mathrm{sVect}(k) \\
\downarrow & & \downarrow & & \downarrow & & \| \\
\mathrm{Chow}(k)_{\mathbb{Q}}/\mathrm{Ker} & \longrightarrow & (\mathrm{Chow}(k)_k/_{-\otimes\mathbb{Q}(1)})/\mathrm{Ker} & \longrightarrow & \mathrm{NHom}(k)_F & \underset{HP_*}{\Longrightarrow} & \mathrm{sVect}(k) \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Num}(k)_k & \longrightarrow & \mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)} & \longrightarrow & \mathrm{NNum}(k)_k & &
\end{array}
$$

where Ker stands for the kernel of the relevant composed horizontal functor. As explained in the proof of Theorem 1.3, the upper horizontal composition corresponds to the functor $s\overline{H_{dR}^*}$ (see (10.8)). Since by hypothesis the standard conjecture $C(Z)$ (and hence the sign conjecture $C^+(Z)$) holds for all smooth projective k-schemes Z, the functors

$$\mathrm{Chow}(k)_k/\mathrm{Ker} \to \mathrm{sVect}(k), \quad \mathrm{Chow}(k)_k/\mathrm{Ker} \to \mathrm{Num}(k)_k$$

satisfy the conditions of the general Proposition 11.7. Proposition 11.7 is clearly functorial on $\mathcal{C}$ and so when applied to the two lower rows of the above commutative diagram, gives rise to sequence

$$\mathrm{Num}^{\dagger}(k)_k \to (\mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)})^{\dagger} \to \mathrm{NNum}^{\dagger}(k)_k. \quad (14.12)$$

By Lemma 4.9 (with $\mathcal{C} = \mathrm{Num}(k)_k$ and $\mathcal{O} = \mathbb{Q}(1)$), we have a canonical $\otimes$-equivalence

$$\mathrm{Num}^\dagger(k)_k/_{-\otimes\mathbb{Q}(1)} \xrightarrow{\sim} (\mathrm{Num}(k)_k/_{-\otimes\mathbb{Q}(1)})^\dagger$$

and so the sequence (14.12) reduces to (14.11). Now, recall from Example 13.3(ii) that $(\underline{\mathrm{Num}}^\dagger(k)_k, w, \mathbb{Q}(1))$ is a neutral Tate triple whose fiber functor is induced from the functor $\overline{H_{dR}^*}$ associated to de Rham cohomology. By Proposition 14.1 we then obtain an induced super-group scheme isomorphism

$$\mathrm{sGal}((\mathrm{Num}^\dagger(k)_k/_{-\otimes\mathbb{Q}(1)})^\natural) \xrightarrow{\sim} \mathrm{Ker}(t : \mathrm{sGal}(\mathrm{Num}^\dagger(k)_k) \twoheadrightarrow \mathbb{G}_\mathrm{m}). \qquad (14.13)$$

On the other hand, since $\mathrm{NNum}^\dagger(k)_k$ is idempotent complete and semisimple (see Theorem 5.6), the functor $R_\mathcal{N}$ in diagram (14.11) gives rise to a faithfully flat morphism

$$\mathrm{sGal}(\mathrm{NNum}^\dagger(k)_k) \twoheadrightarrow \mathrm{sGal}((\mathrm{Num}^\dagger(k)_k/_{-\otimes\mathbb{Q}(1)})^\natural).$$

By combining it with (14.13) we finally obtain the morphism (1.9) of Theorem 1.7. $\qquad\square$

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