Multiplicative genera for noncommutative manifolds?

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Disclaimer: this is a largely speculative talk, meant for an informal discussion session at the workshop "Novel approaches to the finite simple groups" in Banff

Multiplicative Genera of manifolds (Hirzeburch)

• *multiplicative genus*: closed oriented smooth manifolds M; values in commutative unital $\mathbb{Q}$-algebra Λ

$$\phi(M \amalg N) = \phi(M) + \phi(N)$$

$$\phi(M \times N) = \phi(M)\phi(N)$$

$$\phi(\partial M) = 0$$

$\Rightarrow$ depend on *cobordism* class $[M]$

• Oriented cobordism ring $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^n]_{n\geq 1}$ polynomial ring
$\Rightarrow$ genus determined by series

$$\psi(t) = t + \frac{\phi(\mathbb{CP}^2)}{3} t^3 + \frac{\phi(\mathbb{CP}^4)}{5} t^5 + \cdots \in \Lambda[[t]]$$

• Thom: homomorphism $\phi : \Omega_n^{SO} \to \Lambda$ combination of Pontrjagin numbers

Elliptic genus

• multiplicative genus ϕ is *elliptic* if vanishes on $\mathbb{CP}(E)$ projectivized complex vector bundles $E \to M$ over a closed oriented manifold $\Rightarrow$

$$\psi(t) = \int_0^t \frac{du}{\sqrt{1 - 2\delta u^2 + \epsilon u^4}}, \qquad \text{some } \epsilon, \delta \in \Lambda$$

$\Lambda = \mathbb{C}$: signature ($\epsilon = \delta = 1$) $\hat{A}$-genus ($\delta = -1/8$, $\epsilon = 0$)

• Jacobi quartics $y^2 = x^4 - 2\delta x^2 + \epsilon$: as functions of τ modular forms ϵ, δ of level $\Gamma_0(2) \Rightarrow \phi(M)$ polynomial in ϵ, δ, modular form, $\Lambda = M_*(\Gamma_0(2))$

Dirac operator on loop space (Witten)

• X with G action, $F(g) = \mathrm{Tr}_{Ker(D)}(g) - \mathrm{Tr}_{Coker(D)}(g)$ character valued Dirac index: in terms of fixed points X_α component $N = \oplus_\ell N_\ell$ normal bundle $g = e^{\theta P}$ with P acting on N_ℓ as $i\ell$

$$F_\alpha(\theta) = \epsilon_\alpha \langle \hat{A}(M_\alpha)\mathrm{ch}(\sqrt{\det(\otimes_{\ell>0} N_\ell)} \prod_\ell e^{i\theta\ell n_\ell/2} \bigotimes_{\ell>0} \frac{1}{1 - e^{i\ell\theta} N_\ell}), X_\alpha \rangle$$

$(1 - tV)^{-1} = 1 \oplus tV \oplus t^2 S^2 V \oplus \cdots \oplus t^k S^k V \oplus \cdots$
sign ϵ_α (orientation)

• $X = \mathcal{L}(M)$ loop space, M fixed points; normal bundle $\oplus_\ell N_\ell$ each $N_\ell = T = TM$; $n_\ell = d = \dim M$; $\sqrt{\det(\otimes_{\ell>0} N_\ell)}$ choice of spin structure on M

$$F(q) = q^{-d/24} \langle \hat{A}(M) \mathrm{ch}(\otimes_{\ell=1}^\infty S_{q^\ell} T), M \rangle$$

replacing $\prod_{\ell>0} e^{i\theta n_\ell/2}$ with

$$(\prod_{n=1}^\infty q^n)^{d/2} = (q^{\sum_n n})^{d/2} = q^{\zeta(-1)d/2} = q^{-d/24}$$

• $F(q) = \Phi(q)/\eta(q)$, with $\eta(q) = q^{1/24} \prod_{\ell\geq 1}(1-q^\ell)$ Dedekind eta function, and $\Phi(q)$ modular form = level one elliptic genus (assuming $p_1(M) = 0$)

• E. Witten, *The index of the Dirac opeator in loop space*, LNM, Vol.1326 (1988) 161–181.

24-dimensional manifolds and the Monster

- M spin manifold with $p_1(M) = 0 \Rightarrow$ Witten genus Φ_M in $\mathcal{M}_* = \mathbb{Z}[E-4, E-6, \Delta]/(E_4^3 - E_6^2 - 1728\Delta)$ ring of modular forms, $\Delta = q\prod_n (1-q^n)^{24}$
- (Hirzebruch) M as above dim 24: $\Phi_M = \hat{A}(M)\bar{\Delta} + \hat{A}(M, T_{\mathbb{C}})\Delta$ with $\bar{\Delta} = E_4^3 - 744\Delta$
- Question (Hirzebruch): is there a M dim 24, spin, $p_1(M) = 0$, $\hat{A}(M) = 1$, $\hat{A}(M, T_{\mathbb{C}}) = 0$? (that is $\Phi_M = \bar{\Delta}$, or Witten genus j after normalization by η^{24}) Answer: Yes (Hopkins–Mahowald)
- Question (Hirzebruch): is there an action of the monster group $\mathbb{M}$ on such manifold, so that Monster representations (dims related to coeffs of mod form j) from tensor powers of tangent bundle? Not known
- Question: what about a noncommutative manifold?

Noncommutative spin manifolds = Spectral triples

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$, dense domain
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- *even* if $\mathbb{Z}/2$- grading γ on $\mathcal{H}$

$$[\gamma, a] = 0, \ \forall a \in \mathcal{A}, \qquad D\gamma = -\gamma D$$

Main example $(C^\infty(M), L^2(M, S), \partial\!\!\!/_M)$ with chirality γ_5 in 4-dim

• Alain Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995), no. 3, 203–238.

Real structure KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$
antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Commutation: $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$
where $b^0 = Jb^* J^{-1} \qquad \forall b \in \mathcal{A}$
Order one condition:

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$$

Spectral triples in NCG need not be manifolds:

- Quantum groups
- Fractals
- NC tori

• Large classes of NC manifold are deformations of commutative manifolds: Connes–Landi isospectral deformations; quantum groups

• Other classes include "almost commutative" geometries (roughly: bundles of matrix algebras over commutative)

The notion of dimension

For NC spaces: different notions of dimension for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- Metric dimension: growth of eigenvalues of Dirac operator
- KO-dimension (mod 8): sign commutation relations of J, γ, D
- Dimension spectrum: poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$

For manifolds first two agree and third contains usual dim; for NC spaces not same: DimSp $\subset \mathbb{C}$ can have non-integer and non-real points, KO not always metric dim mod 8

Disjoint unions and products

• disjoint union $X = X_1 \amalg X_2$ of two n-dimensional manifolds becomes direct sum: algebra $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, Dirac operator

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

• Product $X_1 \times X_2$ (even case)

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$$

$$\gamma = \gamma_1 \otimes \gamma_2 \qquad J = J_1 \otimes J_2$$

Local index formula: Pontrjagin classes in NCG

$(\mathcal{A}, \mathcal{H}, D)$ even spectral triple: Chern character local formula

$$\phi_n(a_0, \ldots, a_n) = \sum c_{n,k} \mathrm{Res}\mathrm{Tr}(a^0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-n-2|k|})$$

$$c_{n,k} = \frac{(-1)^{|k|} \Gamma(|k| + n/2)}{k!((k_1+1)\cdots(k_1+k_2+\cdots+k_n+n))}$$

Notation: $\nabla(a) = [D^2, a]$ and $a^{(k)} = \nabla^k(a)$
pairing of cyclic cohomology $HC^*(\mathcal{A})$ and K-theory $K_*(\mathcal{A})$

• A. Connes, H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. 5 (1995), 174–243.

Noncommutative manifolds with boundary (Chamseddine–Connes)

• *boundary even*: $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ with $[a,\gamma]=0$ for all $a\in\mathcal{A}$ and $Dom(D)\cap\gamma Dom(D)$ dense in $\mathcal{H}$

• *boundary algebra*: $\partial\mathcal{A}$ quotient $\mathcal{A}/(J\cap J^*)$, two-sided ideal $J=\{a\in\mathcal{A}|aDom(D)\subset\gamma Dom(D)\}$

• *boundary Hilbert space*: $\partial\mathcal{H}$ closure in $\mathcal{H}$ of $D^{-1}KerD_0^*$, with D_0 symmetric operator restricting D to $Dom(D)\cap\gamma Dom(D)$

• action of $\partial\mathcal{A}$ by $a-D^{-2}[D^2,a]$

• *boundary Dirac*: ∂D def on $D^{-1}KerD_0^*$ with $\langle\xi,\partial D\eta\rangle=\langle\xi,D\eta\rangle$ for $\xi\in\partial\mathcal{H}$ and $\eta\in D^{-1}KerD_0^*$; bounded commutators with $\partial\mathcal{A}$

Multiplicative genera for noncommutative manifolds

Λ unital commutative algebra; $\phi(\mathcal{A}, \mathcal{H}, D)$ values in Λ:

- on disjoint unions:

$$\phi((\mathcal{A}_1, \mathcal{H}_1, D_1) \oplus (\mathcal{A}_2, \mathcal{H}_2, D_2)) = \phi(\mathcal{A}_1, \mathcal{H}_1, D_1) + \phi(\mathcal{A}_2, \mathcal{H}_2, D_2)$$

- on products:

$$\phi((\mathcal{A}_1, \mathcal{H}_1, D_1) \otimes (\mathcal{A}_2, \mathcal{H}_2, D_2)) = \phi(\mathcal{A}_1, \mathcal{H}_1, D_1)\phi(\mathcal{A}_2, \mathcal{H}_2, D_2)$$

- on boundaries:

$$\phi(\mathcal{A}, \mathcal{H}, D) = 0 \quad \text{if} \quad (\mathcal{A}, \mathcal{H}, D) = \partial(\mathcal{A}', \mathcal{H}', D')$$

Defined for (finitely summable) spectral triples (up to cobordism)

Question1: what is the right notion of *elliptic*?

• Vanishing on $\mathbb{CP}(E)$ projective bundles of complex vector bundles $E \to M$ in commutative case

• Noncommutative generalizations: (Hilbert modules)
$E \to M$ vector bundle $\Leftrightarrow \mathcal{E}$ finite projective module over $\mathcal{A}$

• $E \mapsto \mathbb{CP}(E)$ projective bundle: projective bundle on $M =$ principal $PU(\mathcal{H})$-bundle (Banach–Steinhaus); isomorphism classes $H^1(M, PU(\mathcal{H})_M)$ sheaf cohomology. Projective bundle $P = \mathbb{CP}(E)$ of a vector bundle E iff Dixmier–Douady class $\delta(P) \in H^3(M, \mathbb{Z})$ is $\delta(P) = 0$

• NCG generalization: continuous trace C^*-algebras have a Dixmier–Douady class

• Is there any room for *modularity*? see next question...

Question 2: cobordism ring and generators?

• Is there a description of cobordism in terms of (ϕ_n) (noncommutative Pontrjagin classes)???

• Note: not by the original Thom argument, which uses embeddings and normal bundles for manifolds

• but... one has bundles (projective modules, Hilbert modules), and morphisms of spectral triples (bimodules with connections) among which some qualify as "embeddings"... parts of the Thom argument go through

• Is there a power series description of genera in NCG???

NCG view of elliptic genus (Jaffe)

• θ-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$:
$\mathrm{Tr}(|D|^{-s})$ not finite but $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$

• JLO cocycle pairs with K-theory $K_0(\mathcal{A})$

$$\tau_n^{JLO}(a_0, \ldots, a_n; g) = \int_{\Sigma_n} \mathrm{Tr}(\gamma U(g) a_0 e^{-s_0 D^2} da_1 e^{-s_1 D^2} \cdots da_n e^{-s_n D^2}) dv$$

$da = [D, a]$, $a \in \mathcal{A}$; simplex $\Sigma_n = \{\sum_j s_j = 1\}$, $dv = ds_0 \cdots ds_n$

• JLO cocycle is a super-KMS-functional generalizing the notion of a Gibbs state

• Elliptic genus as partition function (Jaffe)

$$\mathrm{Tr}_{\mathcal{H}}(\gamma e^{-i\theta J - i\sigma P - \beta H})$$

with Hamiltonian $H = Q^2 - P$, supercharge Q, twisting angle J, translation P

• Supercharge operator Q as Dirac operator of a θ-summable spectral triple (Jaffe, Connes)

• Equivariant index of Dirac operator Q on loop space computed by evaluation of JLO cocycle

• A. Jaffe, *Twist fields, the elliptic genus, and hidden symmetry*, PNAS 97 (2000) 1418–1422.

The loop space of a noncommutative manifold

• $Maps(S^1, X)$ or $\chi : C(X) \to C(S^1)$ hom C^*-algebras

• Kapranov–Vasserot for schemes (infinitesimal loops) $\mathrm{Hom}(A, R[[t]])$

• Points in NCG: not enough characters $\chi : C(X) \to \mathbb{C}$, but lots of states (linear) $\phi : C(X) \to \mathbb{C}$ with positivity $\phi(a^* a) \geq 0$ and $\phi(1) = 1$ (extremal = point measures = points)

• A similar approach for loops? $\mathcal{L}(X) = \{\ell : C(X) \to C(S^1)\}$ linear with some positivity and normalization

• Given spectral triple $(C^\infty(X), L^2(X, S), \not{D}_X)$ use $\mathcal{L}(X)$ to construct a Hilbert bimodule that modifies this spectral triple

Dirac operator on the loop space

• Idea: normal bundle $N = \oplus_{\ell \neq 0} T_\ell$ of M in $\mathcal{L}(M)$ with $T_\ell \simeq TM$

• $\mathcal{T} = T\mathcal{L}(M)$ pullback of TM to loops $\gamma : S^1 \to M$; $\mathcal{E}$ notrivial real line bundle on S^1 and $\hat{\mathcal{T}} = \mathcal{E} \otimes T\mathcal{L}(M)$

$$\hat{\mathcal{T}}|_M = \oplus_{m \in \mathbb{Z}+1/2} q^m T_m$$

$T_m \simeq TM$ and q^m for S^1-action

• spectral triple for $\mathcal{L}(M)$ with $\mathcal{H} = \oplus_m q^m \mathcal{H}_m$, $\mathcal{H}_m = L^2(M, S)$

• twisted Dirac operator on X

$$\not{D}_X \otimes \otimes_{n \geq 1} S^{q^n}(TX_{\mathbb{C}}) \otimes S \otimes \otimes_{n > 0} \Lambda^{q^n}(TX_{\mathbb{C}})$$

• Dirac should give right thing for LG loop groups (Landweber)

• *string structures* on manifolds and spin connections on the loop space

Question 3: a noncommutative space for moonshine?

Why looking for a noncommutative answer?

- Relations between NC spaces from Quantum Statistical Mechanics (GL_2-system) and moonshine: see ongoing work of Jorge Plazas
- Operator algebra approach to CFT (Wassermann, Jones, Longo, Kawahigashi...)
- QSM systems for number fields with phase transitions: CFTs at phase transition? any possible relation to RCFTs with CM of Gukov–Vafa?