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Research

Noncommutative geometry and number theory

Noncommutative geometry, the study of spaces with a not necessarily commutative algebra of coordinates, is a field that has emerged from theoretical physics. In recent years, it has also directed its efforts to arithmetical problems, including the study of the Riemann zeta function. In this article, Matilde Marcolli provides us with some impressions of this emerging field.

Noncommutative geometry is a modern field of mathematics begun by Alain Connes in the early 1980s. It provides powerful tools to treat spaces that are essentially of a quantum nature. Unlike the case of ordinary spaces, their algebra of coordinates is noncommutative, reflecting phenomena like the Heisenberg uncertainty principle in quantum mechanics.

What is especially interesting is the fact that such quantum spaces are abundant in mathematics. One obtains them easily when one considers equivalence relations that are so drastic that they tend to collapse most points together, yet one wishes to retain enough information in the process to be able to do interesting geometry on the resulting space.

In such cases, noncommutative geometry shows that there is a quantum cloud surrounding the classical space, which retains all the essential geometric information, even when the underlying classical space becomes extremely degenerate. It is to this quantum aura that all sophisticated tools of geometry and mathematical analysis, properly reinterpreted, can still be applied.

It has become increasingly evident in recent years that the tools of noncommutative geometry may find new and important applications in number theory, a very different branch of pure mathematics with an ancient and illustrious history. This has happened mostly through a new approach of Connes to the Riemann hypothesis (at present the most famous unsolved problem in mathematics).

Quantum computers

The first instance of such connections between noncommutative geometry and number theory emerged earlier, when Bost and Connes discovered a very interesting noncommutative space with remarkable arithmetic properties. The system it describes consists of quantized optical phases, discretized at different scales. These are essentially the phasors used in modelling quantum computers (see Figure 1). A mechanism that accounts for consistency over scale changes organizes the phasors via a kind of renormalization procedure. This consistency condition imposes the equivalence relation that makes the resulting space noncommutative.

The system obtained in this way has intrinsic dynamics, which makes it evolve in time, and one can consider corresponding thermodynamic equilibrium states at various temperatures. Above a certain critical temperature the distribution of phases is essentially chaotic and there is a unique equilibrium state. At the critical temperature the system undergoes a phase transition with spontaneous symmetry breaking and below critical temperature the system exhibits many different equilibrium states parameterized by arithmetic data.

Especially interesting is what happens at zero temperature. There the arithmetic structure that governs the action of the symmetry group of the system on the extremal ground states is the same one that answers the famous mathematical problem (solved by Gauss) of which regular polygons can be con-

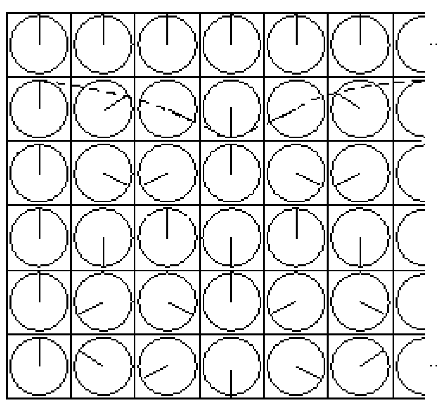

Figure 1 Phase operators: $\mathbf{Z}/6\mathbf{Z}$ discretization

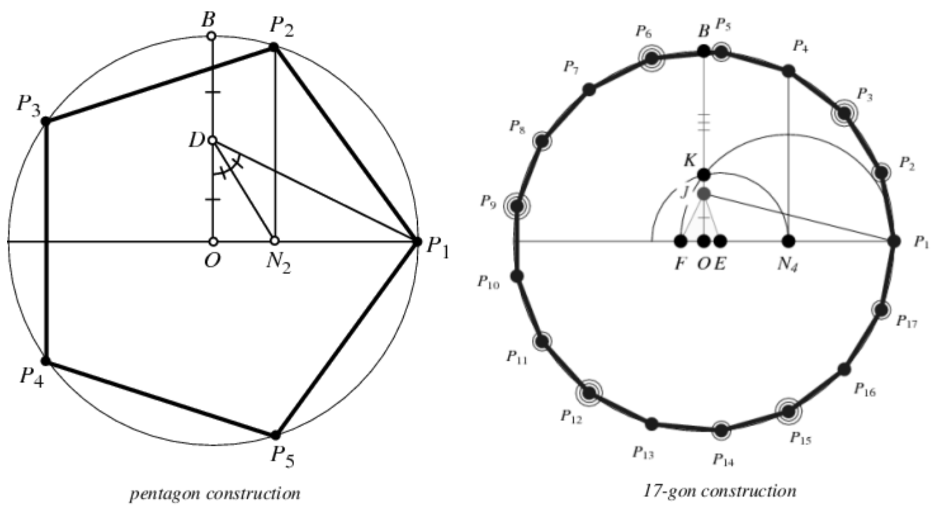

Figure 2 Construction of polygons by ruler and compass

structed using only ruler and compass (see Figure 2).

The crucial feature that allows a solution of this geometric problem is the fact that, in addition to the obvious rotational symmetries of regular polygons, there exists another hidden and much more subtle symmetry coming from the Galois group Gal($\bar{\mathbf{Q}}/\mathbf{Q}$), a very beautiful and still mysterious object, which in this case manifests itself not through the multiplicative action of roots of unity (rotations of the vertices of the polygons) but through the operation of raising them to powers.

Modular curves and non-commutative spaces

Thus, from the example of the Bost Connes noncommutative space, a dictionary emerges that relates the phenomena of spontaneous symmetry breaking in quantum statistical mechanics to the mathematics of Galois theory. Moreover, the partition function of this quantum statistical mechanical system is an object of central interest in number theory, namely the Riemann zeta function (see figure 6). More recently, other results that point to deep connections between noncommutative geometry and number theory appeared in the work of Connes and Moscovici on modular Hecke algebras, which showed that the Rankin–Cohen brackets, an important algebraic structure on modular forms extensively studied years ago by Don Zagier, have a natural interpretation in the language of noncommutative geometry. Modular forms are a class of functions of fundamental importance in many fields of mathematics, especially in number theory and arithmetic geometry. They exhibit elaborate symmetry patterns associated to certain tessellations of the hyperbolic plane (see figure 5).

Figure 3 An example of codimension one foliations

When viewed with the eyes of noncommutative geometry the algebraic structures studied by Zagier appear as a manifestation of a type of symmetry of noncommutative spaces, related to the transverse geometry of codimension one foliations (figure 3), which was investigated extensively in the work of Connes and Moscovici.

The special tessellations of the hyperbolic plane mentioned in relation to modular forms give rise to a family of 2-dimensional surfaces known as the modular curves. Recent work of Manin and Marcolli showed that much of the rich arithmetic structure of the modular curves is captured by a noncommutative space that arises from the tessellation restricted to the infinitely distant horizon of the hyperbolic plane (the bottom horizontal line in figure 5). The fact that the infinite horizon of modular curves hides a noncommutative space was also observed in the work of Connes, Douglas and Schwarz in the context of string theory.

Ongoing work of Connes and Marcolli uncovered the remarkable fact that all the instances listed above of interactions between number theory and noncommutative geometry (Connes' work on the Riemann zeta function, the Bost–Connes system, the modular Hecke algebra and the noncommutative boundary of modular curves) are, in fact, manifestations of the same underlying noncommutative space, namely the space of commensurability classes of **Q**-lattices.

Q-lattices

A lattice consists of arrays of points in a vector space, arranged like atoms in a crystal. For example, the set of points with integer coordinates in the plane is a 2-dimensional lattice. A **Q**-lattice is one such object where one has a way of labelling the points of rational coordinates inside the fundamental cell of the lattice. If each rational point is labelled in a unique way the **Q**-lattice is said to be invertible, while in general one also allows for labellings that miss certain arrays of points while assigning multiple labels to others (see figure 4).

When studying the geometric properties of **Q**-lattices, it is natural to treat as the same object all **Q**-lattices that have the same rational points and where the respective labellings agree whenever both are defined. This determines an equivalence relation on the set of **Q**-lattices. One observes that the identifications produced by this seemingly harmless equivalence relation are in fact drastic enough to give rise to a noncommutative space. On the other hand, if one restricts attention to just invertible **Q**-lattices, these are organized in a classical space. In the case of 2-dimensional lattices, the parameterizing space is the family of all modular curves.

Since **Q**-lattices exist in any dimension, there is in any dimension a corresponding noncommutative space. The Bost–Connes space is just the space of commensurability

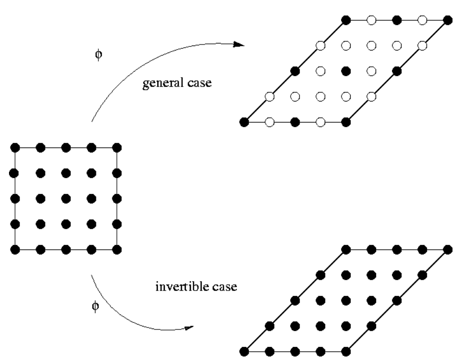

Figure 4 **Q**-lattices

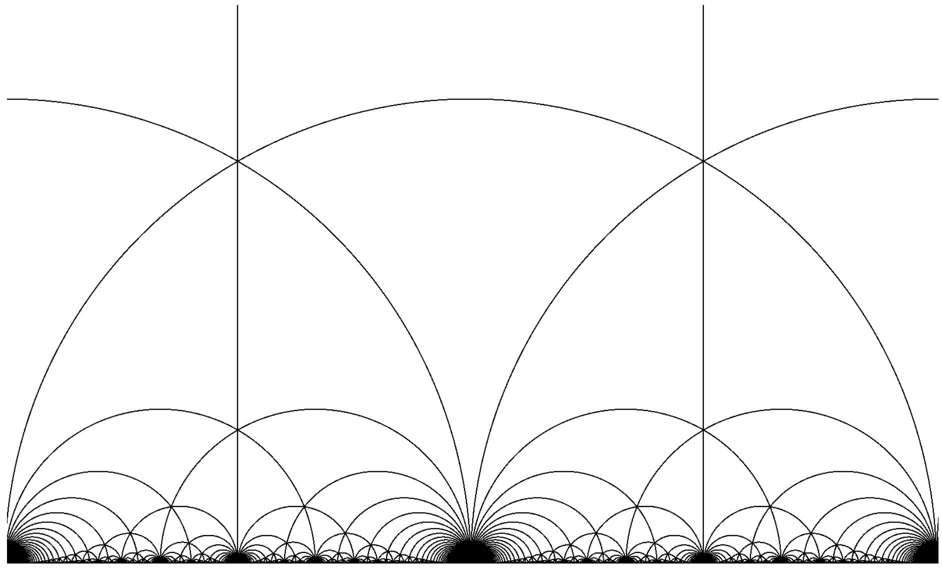

Figure 5 The modular curve

classes of 1-dimensional **Q**-lattices considered up to a scaling factor. The noncommutative space introduced by Connes in the spectral realization of the zeros of the Riemann zeta function (whose position in the plane is the content of the Riemann hypothesis) is the space of commensurability classes of 1-dimensional **Q**-lattices with the scale factor taken into account. The modular Hecke algebra of Connes and Moscovici is a piece of the algebra of coordinates on the space of commensurability classes of 2-dimensional **Q**-lattices and the noncommutative boundary of modular curves is a stratum in this space that accounts for possible degenerations of the 2-dimensional lattice.

The noncommutative space of commensurability classes of 2-dimensional **Q**-lattices up to scale also has a natural time evolution and one can investigate the structure of the corresponding thermodynamic equilibria. At zero temperature this quantum space freezes on the underlying classical space (the family of modular curves) and all quantum fluctuations cease. The extremal states at zero temperature correspond to points on a modular curve. When the temperature rises quantum effects become predominant and the system undergoes a first phase transition where all the different equilibrium states merge, leaving a unique chaotic state. There is then a second critical temperature where the system experiences another phase transition after which no equilibrium state survives.

What acts as a group of symmetries of this quantum mechanical system is the group of all arithmetic symmetries of the modular functions. As in the 1-dimensional case, the induced action on the extremal states at zero temperature is via Galois theory. In this 2-dimensional system, however, not all symmetries act directly on the classical space at zero temperature as they need the more refined structure of the quantum system. Hence one obtains the Galois action at zero temperature by warming up below critical temperature, looking at the full symmetries of the quantum system, and then cooling down again to zero temperature where arithmeticity becomes apparent.

Zeros of the Riemann zeta function

The noncommutative space of commensurability classes of **Q**-lattices with its rich arithmetic structure provides a valuable tool for investigating many related number theoretic questions. For instance, in the spectral realization of zeros of the Riemann zeta function an important question is how to pass consistently to extensions of the field of rational numbers. In the case of imaginary quadratic fields (extensions $\mathbf{Q}(\sqrt{-d})$ of the rational numbers by an imaginary number that is the square root of a negative integer) an analogue of the Bost–Connes quantum statistical mechanical system that has the same properties and the same relation to the Galois theory of abelian extensions was constructed in more recent work of Connes, Marcolli and Ramachandran. The Galois theory of abelian extensions of imaginary quadratic fields is related to the beautiful theory of elliptic curves with complex multiplication and in fact the corresponding quantum statistical mechanical system has a natural formulation in terms of the Tate modules of elliptic curves and the isogeny relation. It can be seen as a specialization of the dynamical system of Connes–Marcolli for **Q**-lattices of rank two, when restricted to those **Q**-lattices that are also 1-dimensional lattices over the field $\mathbf{Q}(\sqrt{-d})$. The construction of Connes–Marcolli was further generalized by Eugene Ha and Frédéric Paugam to a large class of interesting moduli spaces in arithmetic geometry: Shimura varieties, with the Bost–Connes and the Connes–Marcolli systems representing the simplest zero-dimensional and 1-dimensional cases. Benoit Jacob and, using different methods, Consani and Marcolli extended the Bost–Connes construction further to the positive characteristic case of function fields of curves over finite fields. Instead of **Q**-lattices and commensurability, one works in this case, similarly, with Tate modules of rank one Drinfeld modules and isogeny.

An especially interesting and challenging case is that of real quadratic fields $\mathbf{Q}(\sqrt{d})$. Understanding the Galois theory of abelian extensions of such fields is a very important open problem in number theory. A main obstacle comes from the fact that one is missing geometric objects playing in this case the same role that elliptic curves with complex multiplication play in the case of imaginary quadratic fields. In an inspiring and groundbreaking paper, Yuri Manin outlined a striking parallel between the theory of elliptic curves with complex multiplication and the theory of noncommutative tori with real multiplication. This suggests that noncommutative geometry may well provide the missing structure that is needed in this case. There are many challenges implicit in implementing this 'Real Multiplication Program', most importantly the fact that one needs to identify suitable 'coordinate functions for torsion points' on the real multiplication noncommutative tori, analogous to the role that the Weierstrass $\wp$-function plays for elliptic curves. Re-phrased in terms of the quantum statistical mechanical systems of Bost–Connes type, this problem consists of identifying the 'arithmetic elements' in the algebra of the quantum statistical mechanical system associated to the real quadratic field by the general Ha–Paugam construction. Working directly with the noncommu-

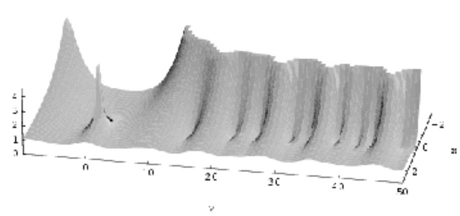

Figure 6 Absolute value of the Riemann zeta function

tative tori, several important advances have been made in the past couple of years, starting with a very important contribution by Polishchuk, who showed that the noncommutative tori with real multiplication have an algebraic model as an algebro-geometric noncommutative space, in addition to their usual analytic model. This algebraic version was related to Manin's quantized theta functions by Marya Vlasenko, while an explicit presentation in terms of modular forms was given by Jorge Plazas. The same algebraic model of noncommutative tori developed by Polishchuk and Polishchuk–Schwarz also provided the basis for a Riemann–Hilbert correspondence for noncommutative tori of Mahanta–van Suijlekom. The analytic model of noncommutative tori can also be used to obtain arithmetic information: Marcolli recently showed that the Shimizu L-function of a lattice in a real quadratic field is naturally obtained from a Lorentzian metric (spectral triple) on the noncommutative torus. The 'Real Multiplication Program' remains a rapidly developing and exciting part of the interaction between noncommutative geometry and number theory.

The Weil proof of the Riemann hypothesis for function fields

Connes' approach to the Riemann hypothesis featured prominently in recent developments in the field, through the work of Connes–Consani–Marcolli on *endomotives* and spectral realizations of L-functions. Abstracting from the class of examples of Bost–Connes–like quantum statistical mechanical systems mentioned above, one can identify a pseudo-abelian category of noncommutative spaces that combines the simplest category of motives, the Artin motives of zero dimensional algebraic varieties, with actions by endomorphisms of abelian semigroups. The algebra of the Bost–Connes system can be seen as an example of a semigroup action on a projective limit of Artin motives, and one obtains a large class of similar examples from self maps of algebraic varieties. These noncommutative spaces have a natural time evolution, induced from a counting measure on the algebraic points of the zero-dimensional varieties, and one can study the associated thermodynamic equilibrium states at varying temperatures. The low temperature equilibrium states provide a good notion of *classical points* of a noncommutative space and the restriction map at the level of algebras that corresponds to the inclusion of the classical points is defined as a morphism in an abelian category of 'noncommutative motives'. The cokernel of this map and its cyclic homology carry a scaling action of the positive real numbers, which is related to the ambiguity in choosing a Hamiltonian for the time evolution. The scaling action on the cyclic homology provide an analogue of the Frobenius action in the characteristic zero case. In the work of Consani–Marcolli on function fields it is shown that this same scaling action in the function field case is indeed obtained from the action of Frobenius (up to a Wick rotation to 'imaginary time'). The work of Connes–Consani–Marcolli shows that the scaling action on the cyclic homology of the cokernel of the restriction map gives a spectral realization of the zeros of the Riemann zeta function (or of L-functions with Grössencharakter) and that a cohomological version of Connes' result on the Weil explicit formula as a trace formula holds on this same cyclic homology. In this formulation the Riemann Hypothesis becomes equivalent to a positivity problem for the trace of certain correspondences on the underlying noncommutative space. This leads to a very suggestive dictionary of analogies between the Weil proof of the Riemann Hypothesis for function fields, which is based on the algebraic geometry of the underlying curve over a finite field, and the noncommutative geometry notions involved in the Connes trace formula. Expanding and developing this dictionary of analogies is another current focus of research in the field.

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