

Quantum field theory and motives

Matilde Marcolli

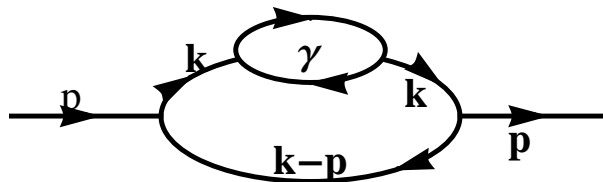
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Based on: joint work with Paolo Aluffi (FSU):
 arXiv:0807.1690, arXiv:0811.2514,
 arXiv:0901.2107, arXiv:0907.3225.

Quantum Fields and Motives (an unlikely match)

- *Feynman diagrams: graphs and integrals*

$$(2\pi)^{-2D} \int \frac{1}{k^4} \frac{1}{(k-p)^2} \frac{1}{(k+\ell)^2} \frac{1}{\ell^2} d^D k d^D \ell$$



Divergences \Rightarrow Renormalization

- *Algebraic varieties and motives*

$\mathcal{V}_{\mathbb{K}}$ smooth proj alg varieties over $\mathbb{K} \Rightarrow$ category of pure motives $\mathcal{M}_{\mathbb{K}}$: $\text{Hom}((X, p, m), (Y, q, n)) = q\text{Corr}_{/\sim}^{m-n}(X, Y)_p$

with $p^2 = p$, $q^2 = q$, $\mathbb{Q}(m) =$ Tate motives

Universal cohomology theory for algebraic varieties

What do they have in common?

Main question: are residues of Feynman integrals periods of mixed Tate motives?

Supporting evidence

- Multiple zeta values from Feynman integral calculations (Broadhurst–Kreimer)
- Parametric Feynman integrals as periods (Bloch–Esnault–Kreimer)
- Graph hypersurfaces and their motives (Belkale–Brosnan)
- Hopf algebras of renormalization (Connes–Kreimer)
- Flat equisingular connections and Galois symmetries (Connes-M.)
- Feynman integrals and Hodge structures (Bloch–Kreimer; M.)

Perturbative QFT in a nutshell

\mathcal{T} = scalar field theory in spacetime dimension D

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Effective action and perturbative expansion (1PI graphs)

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{\Gamma(\phi)}{\#\text{Aut}(\Gamma)}$$

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \hat{\phi}(p_1) \cdots \hat{\phi}(p_N) U_{\mu}^z(\Gamma(p_1, \dots, p_N)) dp_1 \cdots dp_N$$

$$U(\Gamma(p_1, \dots, p_N)) = \int I_{\Gamma}(k_1, \dots, k_{\ell}, p_1, \dots, p_N) d^D k_1 \cdots d^D k_{\ell}$$

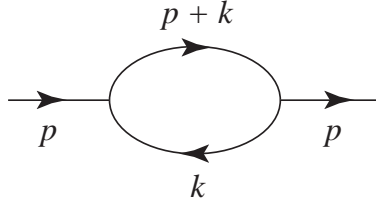
$\ell = b_1(\Gamma)$ loops

Dimensional Regularization: $U_{\mu}^z(\Gamma(p_1, \dots, p_N))$

$$= \int \mu^{z\ell} d^{D-z} k_1 \cdots d^{D-z} k_{\ell} I_{\Gamma}(k_1, \dots, k_{\ell}, p_1, \dots, p_N)$$

(Laurent series in $z \in \Delta^* \subset \mathbb{C}^*$)

Regularization (Dim Reg)



$$\int \frac{1}{k^2 + m^2} \frac{1}{((p+k)^2 + m^2)} d^D k$$

ϕ^3 -theory $D = 4$ divergent

Schwinger parameters

$$\frac{1}{k^2 + m^2} \frac{1}{(p+k)^2 + m^2} = \int_{s>0, t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)} ds dt$$

diagonalize quadratic form in exp

$$-Q(k) = -\lambda((k+xp)^2 + ((x-x^2)p^2 + m^2))$$

with $s = (1-x)\lambda$ and $t = x\lambda \Rightarrow$ Gaussian $q = k + xp$

$$\int e^{-\lambda q^2} d^D q = \pi^{D/2} \lambda^{-D/2}$$

$$\int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2 + \lambda m^2)} \int e^{-\lambda q^2} d^D q \lambda d\lambda dx$$

$$= \pi^{D/2} \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2 + \lambda m^2)} \lambda^{-D/2} \lambda d\lambda dx$$

$$= \pi^{D/2} \Gamma(2 - D/2) \int_0^1 ((x-x^2)p^2 + m^2)^{D/2-2} dx$$

Feynman rules

Construction of $I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N)$:

- Internal lines \Rightarrow propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in \mathcal{L})

$$\sum_{e_i \in E(\Gamma): s(e_i)=v} k_i = 0$$

- Integration over k_i , internal edges

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} d^D k_1 \cdots d^D k_n$$

$$n = \#E_{int}(\Gamma), \quad N = \#E_{ext}(\Gamma)$$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

- Connected graphs: $\Gamma = \cup_{v \in T} \Gamma_v$

$$U(\Gamma_1 \amalg \Gamma_2, p) = U(\Gamma_1, p_1) U(\Gamma_2, p_2)$$

- 1PI graphs:

$$U(\Gamma, p) = \prod_{v \in T} U(\Gamma_v, p_v) \frac{\delta((p_v)_e - (p_{v'})_e)}{q_e((p_v)_e)}$$

Parametric Feynman integrals

- Schwinger parameters $q_1^{-k_1} \dots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1) \dots \Gamma(k_n)} \int_0^\infty \dots \int_0^\infty e^{-(s_1 q_1 + \dots + s_n q_n)} s_1^{k_1-1} \dots s_n^{k_n-1} ds_1 \dots ds_n.$$

- Feynman trick

$$\frac{1}{q_1 \dots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \dots + t_n q_n)^n} dt_1 \dots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t, p)^{n - D\ell/2}}$$

$$\sigma_n = \{t \in \mathbb{R}_+^n \mid \sum_i t_i = 1\}, \text{ vol form } \omega_n$$

- Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik} \eta_{ir}$$

Massless case $m = 0$:

$$V_\Gamma(t, p) = \frac{P_\Gamma(t, p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p, t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (compl of spanning tree plus one edge)

$$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2 \quad \text{with} \quad P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$$

$$\text{for } \sum_{e \in E_{ext}(\Gamma)} p_e = 0 \quad \deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t, p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

stable range $-n + D\ell/2 \geq 0$

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}$$

log divergent $n = D\ell/2$, or massive $m \neq 0$ with $p = 0$

$$\Rightarrow \Psi_\Gamma(t)^{n-D(\ell+1)/2} \omega_n$$

Feynman integrals and periods

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t, p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

Graph hypersurfaces

$$\hat{X}_\Gamma = \{t \in \mathbb{A}^n \mid \Psi_\Gamma(t) = 0\}$$

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \mid \Psi_\Gamma(t) = 0\} \quad \text{deg} = b_1(\Gamma)$$

$$\hat{Y}_\Gamma(p) = \{t \in \mathbb{A}^n \mid P_\Gamma(t, p) = 0\}$$

$$Y_\Gamma(p) = \{t \in \mathbb{P}^{n-1} \mid P_\Gamma(t, p) = 0\} \quad \text{deg} = b_1(\Gamma) + 1$$

Relative cohomology (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \setminus X_\Gamma, \Sigma_n \setminus (\Sigma_n \cap X_\Gamma))$$

$$\Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$$

Realization of mixed Tate motive? $\mathfrak{m}(X, Y)$

(Bloch–Esnault–Kreimer)

Mixed Tate motives $\mathcal{MT}(\mathbb{K})$

- Pure motives \mathcal{M} : smooth projective varieties

$$\mathrm{Hom}((X, p, m), (Y, q, n)) = {}_q\mathrm{Corr}_{/\sim}^{m-n}(X, Y)_p$$

p, q projectors, morphisms alg cycles $\mathrm{codim} = \dim X - m + n$, numerical equivalence

$\mathbb{Q}(1) = \mathbb{L}^{-1}$ Tate motive

$\mathcal{M} =$ abelian category, rigid tensor (Tannakian)

- Mixed motives \mathcal{DM} triangulated category (Voevodsky, Levine, Hanamura)

$$\mathfrak{m}(Y) \rightarrow \mathfrak{m}(X) \rightarrow \mathfrak{m}(X \setminus Y) \rightarrow \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

$\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$

- Examples of mixed Tate: constructed with stratifications and locally trivial fibrations from affine spaces
- Over \mathbb{K} number field: t-structure $\mathcal{MT}(\mathbb{K})$ abelian (Tannakian: $G = U \rtimes \mathbb{G}_m$, prounipotent U)

The Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \setminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Additive invariant $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \setminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

same as assigning

$$\chi : K_0(\mathcal{V}) \rightarrow \mathcal{R}$$

ring homomorphism

\Rightarrow *Universal Euler characteristics*

- Example 1: topological Euler characteristic
- Example 2: Gillet–Soulé:

$K_0(\mathcal{M})$ (abelian category of pure motives: virtual motives)

$$\chi : K_0(\mathcal{V})[\mathbb{L}^{-1}] \rightarrow K_0(\mathcal{M}), \quad \chi(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi(X) = W \cdot (X)$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$

Computing in the Grothendieck group

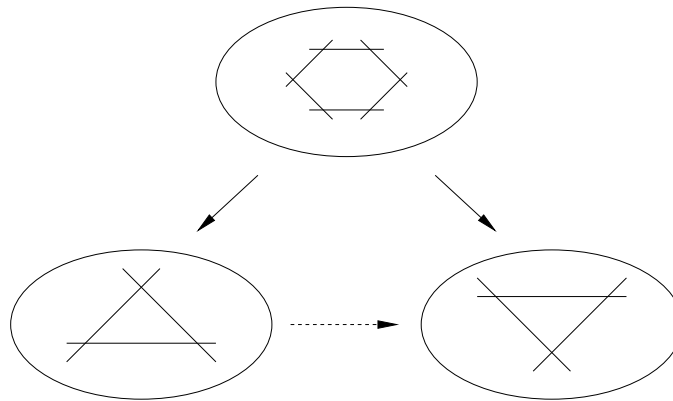
Dual graph and Cremona map

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto \left(\frac{1}{t_1} : \cdots : \frac{1}{t_n} \right)$$

outside \mathcal{S}_n singularities locus of

$$\Sigma_n = \left\{ \prod_i t_i = 0 \right\}$$

ideal $I_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \dots, t_1 t_3 \cdots t_n)$



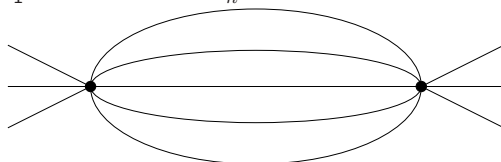
$$\Psi_{\Gamma}(t_1, \dots, t_n) = \left(\prod_e t_e \right) \Psi_{\Gamma^{\vee}}(t_1^{-1}, \dots, t_n^{-1})$$

$$\mathcal{C}(X_{\Gamma} \cap (\mathbb{P}^{n-1} \setminus \Sigma_n)) = X_{\Gamma^{\vee}} \cap (\mathbb{P}^{n-1} \setminus \Sigma_n)$$

isomorphism of X_{Γ} and $X_{\Gamma^{\vee}}$ outside of Σ_n

Example: Banana graphs

$$\Psi_{\Gamma}(t) = t_1 \cdots t_n \left(\frac{1}{t_1} + \cdots + \frac{1}{t_n} \right)$$



Class in the Grothendieck group

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L} - 1)^n - (-1)^n}{\mathbb{L}} - n(\mathbb{L} - 1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive

$X_{\Gamma^v} = \mathcal{L}$ hyperplane in \mathbb{P}^{n-1} ($\Gamma^v =$ polygon)

$$[\mathcal{L} \setminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}$$

$$\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$$

$$X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n$$

$$[\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}$$

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \setminus \Sigma_n]$$

Using Cremona: $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \setminus \Sigma_n]$

$$\Rightarrow \chi(X_{\Gamma_n}) = n + (-1)^n$$

Belkale–Brosnan’s universality

- Classes $[X_\Gamma]$ generate $K_0(\mathcal{V})$ Grothendieck ring of varieties (after localization at \mathbb{L})
- but is the part of the motive involved in the period simpler? a mixed Tate motive?

(Note: Tate part of $K_0(\mathcal{M})$ is just $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$)

Bloch’s sum over graphs (using Cremona)

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\text{Aut}(\Gamma)} \in \mathbb{Z}[\mathbb{L}],$$

Tate motive (though $[X_\Gamma]$ individually need not be)

Feynman rules in algebraic geometry

$U(\Gamma) \in \mathcal{R}$ (comm. ring \mathcal{R} , finite graph Γ)

$$U(\Gamma) = U(\Gamma_1) \cdots U(\Gamma_k) \quad \text{for } \Gamma = \Gamma_1 \amalg \cdots \amalg \Gamma_k$$

$$U(\Gamma) = U(L)^{\#E(T)} \prod_{v \in V(T)} U(\Gamma_v)$$

non-1PI: $\Gamma = \cup_{v \in V(T)} \Gamma_v$

Inverse propagator: $U(L)$ for $L =$ single edge

(Ring homomorphism $U : \mathcal{H}_{CK} \rightarrow \mathcal{R}$ plus choice of $U(L)$:
 $\mathcal{H}_{CK} =$ Connes–Kreimer Hopf algebra)

Algebraic-geometric Feynman rules $\Gamma = \Gamma_1 \amalg \Gamma_2$

$$\mathbb{A}^{n_1+n_2} \setminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \setminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \setminus \hat{X}_{\Gamma_2})$$

$$\Psi_\Gamma(t_1, \dots, t_n) = \Psi_{\Gamma_1}(t_1, \dots, t_{n_1}) \Psi_{\Gamma_2}(t_{n_1+1}, \dots, t_{n_1+n_2})$$

In projective space not product but *join*:

$$\mathbb{P}^{n_1+n_2-1} \setminus X_\Gamma \rightarrow (\mathbb{P}^{n_1-1} \setminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \setminus X_{\Gamma_2})$$

\mathbb{G}_m -bundle (assume Γ_i not a forest)

Ring of immersed conical varieties \mathcal{F}

$V \subset \mathbb{A}^N$ N not fixed, homogeneous ideals (conical),
[V] up to linear changes of coordinates (less than up to
isomorphism)

$$[V \cup W] = [V] + [W] - [V \cap W]$$

$$[V] \cdot [W] = [V \times W]$$

embedded version of Grothendieck ring

- Mod by isomorphisms \Rightarrow maps to $K_0(\mathcal{V})$
- Maps to polynomial invariant

$$I_{CSM} : \mathcal{F} \rightarrow \mathbb{Z}[T]$$

not factoring through Grothendieck group
(characteristic classes of singular varieties)

Algebro-geometric Feynman rules: homomorphisms

$$I : \mathcal{F} \rightarrow \mathcal{R}, \quad \mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

$\Rightarrow I([\mathbb{A}^n \setminus \hat{X}_\Gamma])$ Feynman rule with

$$\mathbb{U}(L) = I([\mathbb{A}^1])$$

Inverse propagator = affine line $[\mathbb{A}^1]$

\Rightarrow Lefschetz motive \mathbb{L}

Universal algebro-geometric Feynman rule

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \setminus \hat{X}_\Gamma] \in \mathcal{F}$$

Motivic = factors through $K_0(\mathcal{V})$

$$[\mathbb{A}^n \setminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \setminus X_\Gamma] \in K_0(\mathcal{V})$$

(if Γ not a forest)

since $[\hat{X}_\Gamma] = (\mathbb{L} - 1)[X_\Gamma] + 1$ affine cone

• Euler characteristic as Feynman rule? Notice

$\chi(\mathbb{A}^n \setminus \hat{X}_\Gamma) = 0$ but nontrivial χ from I_{CSM}

Characteristic classes of singular varieties

- Nonsingular: $c(V) = c(TV) \cap [V]$

$$\int c(TV) \cap [V] = \chi(V)$$

deg of zero dim component = Poincaré–Hopf

- Singular: M.H. Schwartz: radial vector fields;

MacPherson: functoriality $\Rightarrow c_{CSM}(X)$

- Constructible functions $\mathbb{F}(X)$ functor

$$f_*(1_W) = \chi(W \cap f^{-1}(p))$$

- Natural transformation to homology (Chow)

$$c_*(1_X) = c(TX) \cap [X] \text{ for smooth}$$

(Mather classes and local Euler obstructions)

Hypersurfaces with isolated singularities

\Rightarrow Milnor numbers (X_Γ non-isolated singularities)

- Inclusion-exclusion (not isomorphism-invariant)

$$c_{CSM}(X) = c_{CSM}(Y) + c_{CSM}(X \setminus Y)$$

- classes $c_{CSM}(X_\Gamma)$ in ambient \mathbb{P}^{n-1}
(equiv to Eul.char. of iterated hyperplane sections)

Example: banana graphs: $\chi(X_{\Gamma_n}) = \text{top deg term}$

$$c_{CSM}(X_{\Gamma_n}) = ((1 + H)^n - (1 - H)^{n-1} - nH - H^n) \cdot [\mathbb{P}^{n-1}]$$

Feynman rules from CSM classes

$$c_*(1_{\hat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N] \in A(\mathbb{P}^N)$$

natural transformation from constructible function $1_{\hat{X}}$ for $\hat{X} \subset \mathbb{A}^N$ loc closed in \mathbb{P}^N to Chow group $A(\mathbb{P}^N)$

$$G_{\hat{X}}(T) := a_0 + a_1T + \cdots + a_NT^N$$

indep of N , stops at $\dim \hat{X}$; invariant coord. changes;

$$G_{\hat{X} \cup \hat{Y}}(T) = G_{\hat{X}}(T) + G_{\hat{Y}}(T) - G_{\hat{X} \cap \hat{Y}}(T)$$

(from inclusion-exclusion of CSM)

$$I_{CSM}([\hat{X}]) = G_{\hat{X}}(T), \quad I_{CSM} : \mathcal{F} \rightarrow \mathbb{Z}[T]$$

Not easy to see: *ring homomorphism*

$$G_{\hat{X} \times \hat{Y}}(T) = G_{\hat{X}}(T) \cdot G_{\hat{Y}}(T)$$

need CSM classes of joins $J(X, Y) \subset \mathbb{P}^{m+n-1}$

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad (s : t) \in \mathbb{P}^1$$

$\hat{X} \times \hat{Y}$ affine cone over $J(X, Y)$:

$$c_*(1_{J(X, Y)}) = ((f(H) + H^m)(g(H) + H^n) - H^{m+n}) \cap [\mathbb{P}^{m+n-1}]$$

$$c_*(1_X) = H^n f(H) \cap [\mathbb{P}^{n+m-1}], \quad c_*(1_Y) = H^m g(H) \cap [\mathbb{P}^{n+m-1}]$$

CSM Feynman rule:

$$\mathbb{U}_{CSM}(\Gamma) = C_\Gamma(T) = I_{CSM}([\mathbb{A}^n]) - I_{CSM}([\hat{X}_\Gamma])$$

algebraic geometric but not motivic:

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2 + T + 1)$$

$$[\mathbb{A}^n \setminus \hat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V})$$



Properties of $C_\Gamma(T)$:

- $C_\Gamma(T)$ monic of deg n
- $\Gamma = \text{forest} \Rightarrow C_\Gamma(T) = (T+1)^n$
- Inverse propagator $\mathbb{U}_{CSM}(L) = T+1$
- Coeff of T^{n-1} is $n - b_1(\Gamma)$
- $C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \setminus X_\Gamma)$

\Rightarrow modification of $\chi(\mathbb{P}^{n-1} \setminus \hat{X}_\Gamma)$ giving Feynman rule

Deletion–contraction relation

Graph polynomials: Γ with $n \geq 2$ edges, $\deg \Psi_\Gamma = \ell > 0$

$$\Psi_\Gamma = t_e \Psi_{\Gamma \setminus e} + \Psi_{\Gamma/e}$$

$$\Psi_{\Gamma \setminus e} = \frac{\partial \Psi_\Gamma}{\partial t_n} \quad \text{and} \quad \Psi_{\Gamma/e} = \Psi_\Gamma|_{t_n=0}$$

General fact:

$$X = \{\psi = 0\} \subset \mathbb{P}^{n-1}, \quad Y = \{F = 0\} \subset \mathbb{P}^{n-2}$$

$$\psi(t_1, \dots, t_n) = t_n F(t_1, \dots, t_{n-1}) + G(t_1, \dots, t_{n-1})$$

$$\bar{Y} = \text{cone of } Y \text{ in } \mathbb{P}^{n-1}$$

Projection from $(0 : \dots : 0 : 1) \Rightarrow$ isomorphism

$$X \setminus (X \cap \bar{Y}) \xrightarrow{\sim} \mathbb{P}^{n-2} \setminus Y$$

Deletion–contraction formula follows: $\widehat{X}_\Gamma \subset \mathbb{A}^n$

$$[\mathbb{A}^n \setminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \setminus (\widehat{X}_{\Gamma \setminus e} \cap \widehat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma \setminus e}]$$

if e not a bridge or a looping edge

$$[\mathbb{A}^n \setminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma \setminus e}] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma/e}]$$

if e bridge

$$\begin{aligned} [\mathbb{A}^n \setminus \widehat{X}_\Gamma] &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma \setminus e}] \\ &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma/e}] \end{aligned}$$

if e looping edge

Note: intersection $\widehat{X}_{\Gamma \setminus e} \cap \widehat{X}_{\Gamma/e}$ difficult to control motivically: first place where non-Tate contributions will appear

A dichotomy in the Grothendieck ring

(sneak preview of an upcoming paper with P. Aluffi)

Stable birational equivalence:

$$X \times \mathbb{P}^n \simeq Y \times \mathbb{P}^m \text{ (birational)}$$

$\mathbb{Z}[\mathcal{M}]$ = monoid ring of stable birational equivalence classes of smooth projective varieties

Larsen–Lunts:

$$K_0(\mathcal{V}_{\mathbb{C}})/\mathcal{I} = \mathbb{Z}[\mathcal{M}]$$

\mathcal{I} = ideal in $K_0(\mathcal{V}_{\mathbb{C}})$ generated by \mathbb{L}

Using deletion-contraction: graph hypersurfaces are all stably birationally equivalent to points

$$[\mathbb{A}^n \setminus \widehat{X}_{\Gamma}]|_{\mathbb{L}=0} = -[\mathbb{A}^{n-1} \setminus \widehat{X}_{\Gamma \setminus e}]|_{\mathbb{L}=0}$$

reduce to only looping edges $(-1)^n$ or zero

⇒ Implications on periods?

Graphs and determinant hypersurfaces

$$\Upsilon : \mathbb{A}^n \rightarrow \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$

$$[\mathbb{A}^{\ell^2} \setminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1) \Rightarrow \text{mixed Tate}$$

When Υ embedding

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x, p)^{-n+D\ell/2} \omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}$$

If $\hat{\Sigma}_\Gamma$ normal crossings divisor in \mathbb{A}^{ℓ^2}

with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma \Rightarrow$ Question on periods:

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \setminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \setminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \quad \text{mixed Tate?}$$

(motive whose realization is relative cohomology)

Combinatorial conditions for embedding

$$\Upsilon : \mathbb{A}^n \setminus \widehat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \setminus \widehat{\mathcal{D}}_\ell$$

- Closed 2-cell embedded graph:

$$\iota : \Gamma \hookrightarrow S_g$$

$S_g \setminus \Gamma$ union of open disks (faces); closure of each is a disk.

- Two faces have at most one edge in common
- Every edge in the boundary of two faces

Sufficient: Γ 3-edge-connected with closed 2-cell embedding of face width ≥ 3 .

Face width: largest $k \in \mathbb{N}$, every non-contractible simple closed curve in S_g intersects Γ at least k times (∞ for planar).

Note: 2-edge-connected = 1PI; 2-vertex-connected conjecturally implies face width ≥ 2

Identifying the motive $m(X, Y)$

$$\widehat{\Sigma}_\Gamma \subset \widehat{\Sigma}_{\ell, g} \quad (f = \ell - 2g + 1)$$

$$\widehat{\Sigma}_{\ell, g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

$$\begin{cases} x_{ij} = 0 & 1 \leq i < j \leq f - 1 \\ x_{i1} + \cdots + x_{i, f-1} = 0 & 1 \leq i \leq f - 1 \end{cases}$$

$$m(\mathbb{A}^{\ell^2} \setminus \widehat{\mathcal{D}}_\ell, \widehat{\Sigma}_{\ell, g} \setminus (\widehat{\Sigma}_{\ell, g} \cap \widehat{\mathcal{D}}_\ell))$$

$\widehat{\Sigma}_{\ell, g}$ = normal crossings divisor $\Upsilon_\Gamma(\partial\sigma_n) \subset \widehat{\Sigma}_{\ell, g}$
 depends only on $\ell = b_1(\Gamma)$ and $g = \text{min genus of } S_g$

Sufficient conditions for mixed Tate:

- Varieties of frames: mixed Tate?

$$\mathbb{F}(V_1, \dots, V_\ell) := \{(v_1, \dots, v_\ell) \in \mathbb{A}^{\ell^2} \setminus \widehat{\mathcal{D}}_\ell \mid v_k \in V_k\}$$

- Two subspaces: ($d_{12} = \dim(V_1 \cap V_2)$)

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$$

- Three subspaces ($D = \dim(V_1 + V_2 + V_3)$)

$$\begin{aligned} [\mathbb{F}(V_1, V_2, V_3)] &= (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1) \\ &- (\mathbb{L} - 1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1)) \\ &+ (\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3 \end{aligned}$$

Higher: difficult to find suitable induction

Other formulation: $Flag_{\ell, \{d_i, e_i\}}(\{V_i\})$ locus of complete flags $0 \subset E_1 \subset E_2 \subset \dots \subset E_\ell = E$

$$\dim E_i \cap V_i = d_i, \quad \dim E_i \cap V_{i+1} = e_i$$

Are these mixed Tate? (for all choices of d_i, e_i)

$$\begin{aligned} &\mathbb{F}(V_1, \dots, V_\ell) \text{ fibration over } Flag_{\ell, \{d_i, e_i\}}(\{V_i\}): \text{ class } [\mathbb{F}(V_1, \dots, V_\ell)] \\ &= [Flag_{\ell, \{d_i, e_i\}}(\{V_i\})] (\mathbb{L}^{d_1} - 1) (\mathbb{L}^{d_2} - \mathbb{L}^{e_1}) (\mathbb{L}^{d_3} - \mathbb{L}^{e_2}) \dots (\mathbb{L}^{d_r} - \mathbb{L}^{e_{r-1}}) \end{aligned}$$

$Flag_{\ell, \{d_i, e_i\}}(\{V_i\})$ intersection of unions of Schubert cells in flag varieties \Rightarrow Kazhdan–Lusztig?

Removing singularities

- DimReg: local Igusa L-functions (Belkale–Brosnan)

$$I(s) = \int_{\sigma} f(t)^s \omega \Rightarrow \text{Laurent series}$$

coefficients are periods

(log divergent case: more general Bogner–Weinzierl)

- Blowups (Bloch–Esnault–Kreimer) $X_{\Gamma} \cap \Sigma_n$
where singularities can occur: separate via blowups

- (Bloch–Kreimer) monodromies around special divisors of an iterated blowup of \mathbb{P}^{n-1} (toric variety)

- Leray coboundaries (M.M.)

$$D_{\epsilon}(X) = \cup_{s \in \Delta_{\epsilon}^*} X_s$$

$X_s = f^{-1}(s)$, circle bundle $\pi_{\epsilon} : \partial D_{\epsilon}(X) \rightarrow X_{\epsilon}$

integrate around singularities in $\pi_{\epsilon}^{-1}(\sigma \cap X_{\epsilon})$

\Rightarrow Laurent series in ϵ

Regularization and renormalization

Removing divergences from Feynman integrals by adjusting bare parameters in the Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2(1 - \delta Z) + \left(\frac{m^2 - \delta m^2}{2}\right)\phi^2 - \frac{g + \delta g}{6}\phi^3$$

Regularization: replace divergent integral by function with pole

($z \in \mathbb{C}^*$ in DimReg, ϵ deformation of X_Γ , etc.)

Renormalization: consistency over subgraphs

\Rightarrow BPHZ method:

- Preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma)U(\Gamma/\gamma)$$

- Counterterm: projection onto polar part

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$

- Renormalized value:

$$\begin{aligned} R(\Gamma) &= \bar{R}(\Gamma) + C(\Gamma) \\ &= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma)U(\Gamma/\gamma) \end{aligned}$$

Hopf algebra structures (Connes–Kreimer)

$\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depend on theory $\mathcal{L}(\phi)$)

Free commutative algebra in generators
 Γ 1PI Feynman graphs

Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom):
Ward identities as Hopf ideals

Connes–Kreimer theory

- \mathcal{H} dual to affine group scheme G (diffeomorphisms)

- $G(\mathbb{C})$ pro-unipotent Lie group \Rightarrow

$$\gamma(z) = \gamma_-(z)^{-1} \gamma_+(z)$$

Birkhoff factorization of loops exists

- Recursive formula for Birkhoff = BPHZ

- loop = $\phi \in \text{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$
(germs of meromorphic functions)

- Feynman integral $U(\Gamma) = \phi(\Gamma)$
counterterms $C(\Gamma) = \phi_-(\Gamma)$
renormalized value $R(\Gamma) = \phi_+(\Gamma)|_{z=0}$

Algebro-geometric Feynman rules: $\mathbb{U}(\Gamma)$ gives

$$\phi : \mathcal{H} \rightarrow \mathcal{R}, \quad \phi(\Gamma) = \mathbb{U}(\Gamma)$$

algebra homomorphism

Birkhoff factorization works whenever \mathcal{R} has a Rota–Baxter structure of weight $\lambda = -1$ (Ebrahimi-Fard, Guo, Kreimer)

Rota–Baxter weight λ : \exists linear map \mathfrak{P} on \mathcal{R} such that

$$\mathfrak{P}(X)\mathfrak{P}(Y) = \mathfrak{P}(X\mathfrak{P}(Y)) + \mathfrak{P}(\mathfrak{P}(X)Y) + \lambda\mathfrak{P}(XY)$$

For Laurent series: $\mathfrak{P} =$ polar part projection: in CK recursive formula

$$\phi_{-}(X) = -\mathfrak{P}(\phi(X) + \sum \phi_{-}(X')\phi(X''))$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$ gives $\phi_{-}(XY) = \phi_{-}(X)\phi_{-}(Y)$

Question: Is there an interesting Rota–Baxter structure on $K_0(\mathcal{V})$ or on \mathcal{F} ?

Renormalization and motivic Galois theory

(A. Connes–M.M. 2004)

Compare renormalization and motives
by comparing Tannakian categories

- Counterterms as iterated integrals
(’t Hooft–Gross relations)
- Solutions of irregular singular differential equations (flat equisingular connections)
- Flat equisingular vector bundles form a neutral Tannakian category \mathcal{E}
- Free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$

$$\mathcal{E} \simeq \text{Rep}_{\mathbb{U}^*}, \quad \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

$$\mathbb{U} = \text{Hom}(\mathcal{H}_{\mathbb{U}}, -), \quad \text{with } \mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$$

- Motivic Galois group (Deligne–Goncharov)

$$\mathbb{U}^* \simeq \text{Gal}(\mathcal{M}_S)$$

\mathcal{M}_S mixed Tate motives on $S = \text{Spec}(\mathbb{Z}[i][1/2])$

Question: Why $\text{Spec}(\mathbb{Z}[i][1/2])$ from X_Γ ?
Equisingular (irregular singular) connections and
Hodge structures (regular singular)?

Additional considerations: (M.M. 2008)

What does DimReg mean geometrically?

A tentative motivic approach: Kummer motive

$$M = [u : \mathbb{Z} \rightarrow \mathbb{G}_m] \in \text{Ext}_{\mathcal{DM}(\mathbb{K})}^1(\mathbb{Q}(0), \mathbb{Q}(1))$$

with $u(1) = q \in \mathbb{K}^*$ and period matrix

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}$$

Kummer extension of Tate sheaves

$$\mathcal{K} \in \text{Ext}_{\mathcal{DM}(\mathbb{G}_m)}^1(\mathcal{Q}_{\mathbb{G}_m}(0), \mathcal{Q}_{\mathbb{G}_m}(1))$$

$$\mathcal{Q}_{\mathbb{G}_m}(1) \rightarrow \mathcal{K} \rightarrow \mathcal{Q}_{\mathbb{G}_m}(0) \rightarrow \mathcal{Q}_{\mathbb{G}_m}(1)[1]$$

Logarithmic motives $\text{Log}^n = \text{Sym}^n(\mathcal{K})$

$$\text{Log}^\infty = \varprojlim_n \text{Log}^n$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots \\ \log(s) & (2\pi i) & 0 & \cdots & 0 & \cdots \\ \frac{\log^2(s)}{2!} & (2\pi i) \log(s) & (2\pi i)^2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \frac{\log^n(s)}{n!} & (2\pi i) \frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^2 \frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^{n-1} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots \end{pmatrix}$$

Graph polynomials and motivic sheaves M_Γ

$$(\Psi_\Gamma : \mathbb{A}^n \setminus \hat{X}_\Gamma \rightarrow \mathbb{G}_m, \hat{\Sigma}_n \setminus \hat{X}_\Gamma \cap \hat{\Sigma}_n, n-1, n-1)$$

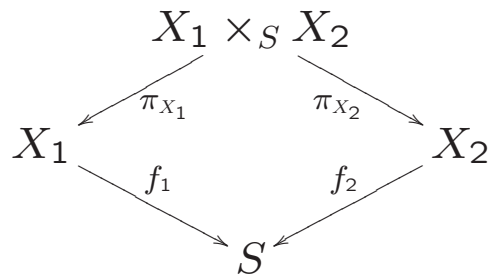
Arapura's category of motivic sheaves $(f : X \rightarrow S, Y, i, w)$

DimReg = product $M_\Gamma \times \text{Log}^\infty$ fibered product

$$(X_1 \times_S X_2 \rightarrow S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2, i_1 + i_2, w_1 + w_2)$$

$$\int \pi_{X_1}^*(\omega) \wedge \pi_{X_2}^*(\eta) = \int \omega \wedge f_1^*(f_2)_*(\eta)$$

on $\sigma_1 \times_S \sigma_2$ for $\sigma_i \subset X_i, \partial\sigma_i \subset Y_i$



DimReg integral $\int_\sigma \Psi_\Gamma^z \alpha$ period on $M_\Gamma \times \text{Log}^\infty$

NCG explanation of DimReg in A. Connes, M.M. *Anomalies, Dimensional Regularization and noncommutative geometry*, unpublished manuscript, 2005, available at www.its.caltech.edu/~matilde/work.html