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Motives and periods in Bianchi IX gravity models

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Abstract We show that, when considering the anisotropic scaling factors and their derivatives as affine variables, the coefficients of the heat-kernel expansion of the Dirac–Laplacian on $SU(2)$ Bianchi IX metrics are algebro-geometric periods of motives of complements in affine spaces of unions of quadrics and hyperplanes. We show that the motives are mixed Tate and we provide an explicit computation of their Grothendieck classes.

Keywords Bianchi IX gravity model · Spectral action · Dirac–Laplacian · Heat kernel expansion · Periods · Motives

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1 Introduction

In this paper we continue our investigation of arithmetic structures arising in models of Euclidean gravity based on the spectral action functional of [4]. More specifically, here by "arithmetic structures" we mean the occurrence of algebro-geometric periods of mixed motives of algebraic varieties defined over number fields. In [11] we showed that the heat-kernel Seeley–deWitt coefficients for the Dirac–Laplacian of the Robertson–Walker metrics can be expressed as periods of mixed Tate motives given by affine complements of unions of quadrics and hyperplanes. In the present paper, we extend the result of [11] on the homogeneous and isotropic Robertson–Walker metrics to the case of the homogeneous but nonisotropic Bianchi IX metrics. Although the argument used in [11] does not immediately apply to the anisotropic case, we provide a different parameterization of the integrals computing the Seeley–deWitt coefficients, for which we can derive a very similar statement about expressing these integrals as periods of certain mixed Tate motives given by complements of unions of quadrics and hyperplanes.

The occurrence of arithmetic structures involving periods and motives has been a focus of investigation in theoretical physics over the past decade. In particular, interesting motivic structures have been uncovered in the context of perturbative quantum field theory (see [20] for an introductory overview). Some examples of occurrences of motives and periods in models of quantum gravity have also been identified, see for example [15]. However, the extent to which algebro-geometric and arithmetic structures play a role in models of gravity remains very much open to investigation. In this and other related papers [7,8,11], we began a systematic study of the role of arithmetic structures in models of gravity based on the spectral action functional. Our focus in this paper is on the case of the Euclidean $SU(2)$-Bianchi IX models of gravity.

The Bianchi IX metrics play an important role in Euclidean quantum gravity and quantum cosmology in the form of minisuperspace models in Hartle–Hawking gravity, see [9]. In view of a similar approach to quantum cosmology based on the spectral action, currently being developed (see [21]), it is interesting to investigate what role of arithmetic structures will play in such gravity models. The Bianchi IX metrics are closely related to the mixmaster cosmological models of [13]. These have been widely studied (see for instance [6,22,24]) and are known to have very interesting relations for number theory, see [16–19]. In [7] we proved a rationality result for the Seeley–deWitt coefficients of the Bianchi IX metrics, which generalizes an analogous rationality result for the Robertson–Walker case conjectured in [3] and proved in [10]. In [8] we proved that, in the case of the two-parameter family of [1] of Bianchi IX gravitational instantons, the Seeley–deWitt coefficients for the Dirac–Laplacian are vector-valued modular forms. We expect that, in addition to these occurrences of motives, periods, and modular forms, a broader range of interesting relations between gravity models based on spectral action and heat kernel and number theory remains to be uncovered.

2 Bianchi IX metrics and Dirac operators

We consider here $SU(2)$-Bianchi IX metrics of the form

$$ds^2 = w_1(t)\, w_2(t)\, w_3(t)\, dt^2 + \frac{w_2(t)\, w_3(t)}{w_1(t)}\, \sigma_1^2 + \frac{w_3(t)\, w_1(t)}{w_2(t)}\, \sigma_2^2 + \frac{w_1(t)\, w_2(t)}{w_3(t)}\, \sigma_3^2, \tag{2.1}$$

where the σ_i are left-invariant 1-forms on $SU(2)$-orbits satisfying the relations

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \qquad d\sigma_2 = \sigma_3 \wedge \sigma_1, \qquad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$

This metric can be written locally as $ds^2 = \sum g_{\mu\nu}\, dx^\mu dx^\nu$, in the set of local coordinates $x = (x^\mu)_{\mu=1,\ldots,4} = (t, \eta, \phi, \psi)$, where the 3-dimensional sphere $\mathbb{S}^3 \simeq SU(2)$ is parameterized by the map

$$(\eta, \phi, \psi) \mapsto \left(\cos(\eta/2)\, e^{i(\phi+\psi)/2}, \sin(\eta/2)\, e^{i(\phi-\psi)/2}\right).$$

Here the parameters have the ranges $0 \le \eta < \pi, 0 \le \phi < 2\pi, 0 \le \psi < 4\pi$. The local formula of the Dirac operator D of the metric in this coordinate system and its pseudodifferential symbol σ_D can be computed as in [7].

We recall that given a spin bundle S on a Riemannian manifold M, the Dirac operator D is a differential operator of order 1 acting on the smooth spinors (sections of S) defined as the composition of the following maps:

$$D = c \circ \# \circ \nabla^S : C^\infty(S) \xrightarrow{\nabla^S} C^\infty(T^*M \otimes S) \xrightarrow{\#} C^\infty(TM \otimes S) \xrightarrow{c} C^\infty(S).$$

Here, the spin connection ∇^S is obtained by lifting the Levi-Civita connection from the tangent bundle to S, the musical isomorphism # identifies the cotangent and tangent bundles, and c denotes the action of the Clifford algebra of the metric on the spinors. Writing the formula for D, one finds that if $\{\theta_\mu\}$ is a local orthonormal frame, then

$$D = \sum_\mu \theta_\mu \nabla^S_{\theta_\mu},$$

in the local coordinates.

By using the local formula for D one can derive its pseudodifferential symbol $\sigma(D)(x, \xi)$. That is, one can locally write the action of D on a spinor $\mathfrak{s}$ as

$$\begin{aligned} D\mathfrak{s}(x) &= (2\pi)^{-\dim(M)/2} \int e^{i\, x\cdot\xi}\, \sigma(D)(x,\xi)\, \hat{\mathfrak{s}}(\xi)\, d\xi \\ &= (2\pi)^{-\dim(M)} \int\int e^{i\,(x-y)\cdot\xi}\, \sigma(D)(x,\xi)\, \mathfrak{s}(y)\, dy\, d\xi, \end{aligned} \tag{2.2}$$

where $\hat{\mathfrak{s}}$ denotes the component-wise Fourier transform of $\mathfrak{s}$. In this formula ξ is in fact an element of the cotangent fiber at the point x, which is identified with the Euclidean space of the same dimension as the manifold M. The Fourier transform is understood here in the usual sense of pseudodifferential operators (ΨDOs) on manifolds. The ΨDOs on manifolds are defined in terms of local coordinates, and invariance of results with respect to coordinate changes is proved (see [26]). Coordinate-free approaches have also been proposed, see for instance [23].

The pseudodifferential symbol σ_D of the Dirac operator D of the Bianchi IX metric given by (2.1) was derived in [7] by explicit calculations. The result is that

$$\sigma(D)(x,\xi) = q_1(x,\xi) + q_0(x,\xi),$$

where

$$\begin{aligned} q_1(x,\xi) = &-\frac{i\gamma^2\sqrt{w_1}\,(\csc(\eta)\cos(\psi)\,(\xi_4\cos(\eta)-\xi_3)+\xi_2\sin(\psi))}{\sqrt{w_2}\sqrt{w_3}} \\ &+\frac{i\gamma^3\sqrt{w_2}\,(\sin(\psi)\,(\xi_3\csc(\eta)-\xi_4\cot(\eta))+\xi_2\cos(\psi))}{\sqrt{w_1}\sqrt{w_3}} \\ &+\frac{i\gamma^1\xi_1}{\sqrt{w_1}\sqrt{w_2}\sqrt{w_3}}+\frac{i\gamma^4\xi_4\sqrt{w_3}}{\sqrt{w_1}\sqrt{w_2}}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} q_0(x,\xi) = &\frac{1}{4\sqrt{w_1w_2w_3}}\left(\frac{w_1^{'}}{w_1}+\frac{w_2^{'}}{w_2}+\frac{w_3^{'}}{w_3}\right)\gamma^1 \\ &-\frac{\sqrt{w_1w_2w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4. \end{aligned}$$

Here the γ^i are 4×4 matrices such that $(\gamma^i)^2 = -I$ and $\gamma^i\gamma^j + \gamma^j\gamma^i = 0$ for $i \neq j$.

Correspondingly, for the Dirac–Laplacian D^2 we have

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

where the homogeneous terms are given by

$$\begin{aligned} p_2(x,\xi) &= q_1(x,\xi)^2, \\ p_1(x,\xi) &= q_0(x,\xi)\,q_1(x,\xi) + q_1(x,\xi)\,q_0(x,\xi) \\ &\quad + \sum_{j=1}^{4}\left(-i(\partial_{\xi_j}q_1)(x,\xi)\,(\partial_{x_j}q_1)(x,\xi)\right), \\ p_0(x,\xi) &= q_0(x,\xi)^2 + \sum_{j=1}^{4}\left(-i(\partial_{\xi_j}q_1)(x,\xi)\,(\partial_{x_j}q_0)(x,\xi)\right). \end{aligned} \tag{2.4}$$

In particular, for later use, note that we can write the degree-two homogeneous term in the form

$$p_2(x,\xi) = \left(\sum_{\mu,\nu=1}^{4} g^{\mu\nu}\xi_\mu\xi_\nu\right) I,$$

where the matrix $(g^{\mu\nu})$ is the inverse of the symmetric matrix $(g_{\mu\nu})$ formed by the components of the metric tensor.

3 Seeley–deWitt coefficients and periods

The spectral action functional of Euclidean gravity, introduced in [4], is defined as a trace $\mathrm{Tr}(f(D/\Lambda))$ of the Dirac operator regularized by an even rapidly decaying function f approximating a cutoff function on the Dirac spectrum, with Λ an energy scale. It can be viewed as a modified gravity model, since the leading terms in the large Λ expansion include the Einstein–Hilbert action of gravity with cosmological term, as well as some higher derivative terms that include conformal gravity and Gauss–Bonnet gravity. Overviews of applications of the spectral action functional to cosmology and particle physics can be found in [21] and [29].

The Seeley–deWitt coefficients $a_{2n}(D^2)$ appearing in the heat-kernel expansion of the Dirac–Laplacian

$$\mathrm{Tr}(e^{-\tau D^2}) \sim_{\tau\to 0^+} \tau^{-\dim(M)/2}\sum_{n=0}^{\infty} a_{2n}(D^2)\tau^n,$$

determine the coefficients of the large energy asymptotic expansion of the spectral action functional, see [4] and §1 of [5] for more details. The meaning of this expansion is that for any nonnegative integer N and for small positive τ we have

$$\mathrm{Tr}(e^{-\tau D^2}) = \tau^{-\dim(M)/2}\sum_{n=0}^{N} a_{2n}(D^2)\tau^n + O(\tau^{-\dim(M)/2+N+1}).$$

Thus, our approach to investigating the arithmetic properties of the spectral action models of gravity is based on identifying arithmetic structures in the Seeley–deWitt coefficients of the heat-kernel expansion of the Dirac–Laplacian.

3.1 The Seeley–deWitt coefficients as residues

For any $n \in \mathbb{Z}_{\geq 1}$, the Seeley–deWitt coefficients a_{2n} can be computed as a noncommutative residue (see [7])

$$a_{2n} = \frac{1}{32\,\pi^{n+3}}\mathrm{Res}(\Delta_{2n}^{-1}), \tag{3.1}$$

where

$$\Delta_{2n} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^{2n-2}},$$

with $\Delta_{\mathbb{T}^{2n-2}}$ the Laplacian of the flat metric on an auxiliary $(2n-2)$-dimensional torus $\mathbb{T}^{2n-2} = (\mathbb{R}/\mathbb{Z})^{2n-2}$. Since the operator Δ_{2n} is acting on the smooth sections of a vector bundle on a (2n+2)-dimensional manifold, in order to calculate $\mathrm{Res}(\Delta_{2n}^{-1})$, we need the term that is positively homogeneous of order $-2n-2$ in the asymptotic expansion of the symbol of Δ_{2n}^{-1}. We write

$$\sigma(\Delta_{2n}^{-1})(x,\xi) \sim_{\xi\to\infty} \sum_{m=-\infty}^{-2} \sigma_m(\Delta_{2n}^{-1})(x,\xi),$$

where each $\sigma_m(\Delta_{2n}^{-1})$ is (positively) homogeneous of order m in ξ.

By definition (see [31,32])

$$\mathrm{Res}\left(\Delta_{2n}^{-1}\right) = \int_{M\times\mathbb{T}^{2n-2}} \left(\int_{|\xi|=1} \mathrm{tr}\left(\sigma_{-2n-2}(x,\xi)\right) |\sigma_{\xi,\,2n+1}| \right) |dx^1 \wedge \cdots \wedge dx^{2n+2}|, \tag{3.2}$$

in which $\sigma_{\xi,\,2n+1}$ is the volume form of the unit sphere $|\xi| = 1$ in the cotangent fiber $\mathbb{R}^{2n+2} \simeq T_x^*(M \times \mathbb{T}^{2n-2})$, given by

$$\sigma_{\xi,\,2n+1} = \sum_{j=1}^{2n+2} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{2n+2}. \tag{3.3}$$

Remark 3.1 Because of the homogeneity degree of $\sigma_{-2n-2}(x,\xi)$ in (3.2) and the Stokes theorem, the integration over the sphere $|\xi| = 1$ can be replaced with integration over the unit sphere of the metric or any other similar locus that is homologous to the sphere as a closed cycle, see Proposition 7.3 on page 265 of [12].

The $\sigma_m(\Delta_{2n}^{-1})$ satisfy the recursive relations (see [7])

$$\sigma_{-2}(\Delta_{2n}^{-1})\,(x,\xi) = \left(p_2(x,\xi_1,\ldots,\xi_4) + \left(\xi_5^2 + \cdots + \xi_{2n+2}^2\right) I \right)^{-1}, \tag{3.4}$$

and, for $m \le -3$,

$$\sigma_m(\Delta_{2n}^{-1})\,(x,\xi) = -\left(\sum_{\substack{\alpha_1,\alpha_2,\alpha_4\in\mathbb{Z}_{\ge 0}\\ m<j\le -2,\quad 0\le k\le 2\\ j-\alpha_1-\alpha_2-\alpha_4+k=m+2}} \frac{(-i)^{\alpha_1+\alpha_2+\alpha_4}}{\alpha_1!\,\alpha_2!\,\alpha_4!} \left(\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_4}^{\alpha_4}\sigma_j(\Delta_{2n}^{-1})\right)\left(\partial_t^{\alpha_1}\partial_\eta^{\alpha_2}\partial_\psi^{\alpha_4} p_k\right) \right) \sigma_{-2}(\Delta_{2n}^{-1}). \tag{3.5}$$

Note that in this expression we have considered the fact that the symbol of the Dirac operator D given by (2.3) is independent of the coordinate ϕ.

3.2 Seeley–deWitt coefficients as period integrals

We focus here on the Seeley–deWitt coefficient before time integration, treating the anisotropy coefficients w_i and their derivatives as affine variables. We show that for algebraic values of these variables the resulting coefficient is a period integral in the algebro-geometric sense (see [14]), that is, an integral of an algebraic differential form on a semi-algebraic set in an algebraic variety.

Remark 3.2 In the following we use the notation α_{2n} for the Seeley–deWitt coefficient prior to integration in the time variable, namely

$$a_{2n} = \int \alpha_{2n}(t)\, dt, \tag{3.6}$$

where the t-dependence of α_{2n} is through the cosmic expansion factors (anisotropy coefficients) $w_i(t)$ of the Bianchi IX metric, for $i = 1, 2, 3$, and their derivatives,

$$\alpha_{2n}(t) = \alpha_{2n}(w_i(t), w_i'(t), w_i''(t), \ldots, w_i^{(2n)}(t)). \tag{3.7}$$

Proposition 3.3 *Introducing new variables*

$$\begin{aligned} W_1 &= \frac{1}{\sqrt{w_1(t)}\sqrt{w_2(t)}\sqrt{w_3(t)}}, \quad W_2 = -\frac{\sqrt{w_1(t)}}{\sqrt{w_2(t)}\sqrt{w_3(t)}}, \\ W_3 &= \frac{\sqrt{w_2(t)}}{\sqrt{w_1(t)}\sqrt{w_3(t)}}, \quad W_4 = \frac{\sqrt{w_3(t)}}{\sqrt{w_1(t)}\sqrt{w_2(t)}}, \end{aligned} \tag{3.8}$$

and the change of coordinates

$$\begin{aligned} \zeta_1 &= \xi_1, \\ \zeta_2 &= \xi_4 \cot(\eta)\cos(\psi) - \xi_3 \csc(\eta)\cos(\psi) + \xi_2 \sin(\psi), \\ \zeta_3 &= -\xi_4 \cot(\eta)\sin(\psi) + \xi_3 \csc(\eta)\sin(\psi) + \xi_2 \cos(\psi) \\ \zeta_4 &= \xi_4, \qquad \zeta_5 = \xi_5, \quad \ldots \quad \zeta_{2n+2} = \xi_{2n+2}, \end{aligned}$$

the expression $tr\,(\sigma_{-2n-2})$ *is given by*

$$tr\,(\sigma_{-2n-2}) = \sum_{j=1}^{M_n} \Bigg\{ c_{j,2n}\, (\sin\eta)^{\beta_{0,1,j}} (\cos\eta)^{\beta_{0,2,j}}\, (\sin\psi)^{\beta_{1,1,j}} (\cos\psi)^{\beta_{1,2,j}} \\ \frac{\zeta_1^{\beta_{1,j}} \zeta_2^{\beta_{2,j}} \cdots \zeta_{2n+2}^{\beta_{2n+2,j}}}{Q_{W,2n}^{\rho_{j,2n}}} \prod_{i=1}^{3} \omega_{i,0}^{k_{i,0,j}} \omega_{i,1}^{k_{i,1,j}} \cdots \omega_{i,2n}^{k_{i,2n,j}} \Bigg\}, \tag{3.9}$$

where

$$c_{j,2n} \in \mathbb{Q},$$
$$\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j}, k_{i,0,j} \in \mathbb{Z},$$
$$\beta_{1,j}, \dots, \beta_{2n+2,j}, \rho_{j,2n}, k_{i,1,j}, \dots, k_{i,2n,j} \in \mathbb{Z}_{\geq 0},$$

where

$$Q_{W,2n}(\zeta_1, \dots, \zeta_{2n+2}) = W_1^2\zeta_1^2 + W_2^2\zeta_2^2 + W_3^2\zeta_3^2 + W_4^2\zeta_4^2 + \zeta_5^2 + \cdots + \zeta_{2n+2}^2, \tag{3.10}$$

with the variables $\omega_{i,j}$ associated with the cosmic expansion factors $w_1(t)$, $w_2(t)$, $w_3(t)$ given by

$$\omega_{i,0} = w_i(t), \qquad \omega_{i,1} = w_i'(t), \qquad \dots \qquad \omega_{i,2n} = w_i^{(2n)}(t). \tag{3.11}$$

Proof This is a direct consequence of (3.4), (3.5), the explicit formulas provided in [7] for the homogeneous symbols p_2, p_1, p_0 (which were calculated using (2.4)), and the fact that

$$p_2(x, \xi_1, \dots, \xi_4) + \xi_5^2 + \cdots + \xi_{2n+2}^2 = Q_{W,2n}(\zeta_1, \dots, \zeta_{2n+2}).$$

□

We can then compute the Seeley–deWitt coefficient α_{2n} of (3.6) as follows.

Proposition 3.4 *The Seeley–deWitt coefficient is given by the integral*

$$\alpha_{2n} = \frac{1}{\pi^{n+2}} \int_0^{\pi/2} \sin(\eta)\, d\eta \int_0^{\pi/2} d\psi \int_{\sum_{i=1}^{2n+2} \zeta_i^2 = 1} tr\,(\sigma_{-2n-2})\, \sigma_{\zeta,\,2n+1}. \tag{3.12}$$

Proof By (3.2) and Remark 3.1 we have

$$\begin{aligned}\alpha_{2n} &= \frac{1}{32\,\pi^{n+3}} \int_0^{\pi} d\eta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \int_{|\xi|_g = 1} \mathrm{tr}\,(\sigma_{-2n-2})\, \sigma_{\xi,\,2n+1} \\ &= \frac{1}{\pi^{n+2}} \int_0^{\pi/2} d\eta \int_0^{\pi/2} d\psi \int_{|\xi|_g = 1} \mathrm{tr}\,(\sigma_{-2n-2})\, \sigma_{\xi,\,2n+1},\end{aligned} \tag{3.13}$$

where $|\xi|_g = \sum_{\mu,\nu=1}^4 g^{\mu\nu}\xi_\mu\xi_\nu + \xi_5^2 + \cdots + \xi_{2n+2}^2$. Note that for the second identity in (3.13), we used the fact that

$$\frac{1}{\sin(\eta)\, w_1(t)\, w_2(t)\, w_3(t)} \int_{|\xi|_g = 1} \mathrm{tr}\,(\sigma_{-2n-2})\ \sigma_{\xi,\,2n+1}$$

is independent of the variables η, ϕ, ψ. This fact is indeed associated with the symmetries of the Bianchi IX metric and was proved in [7]. Next observe that the sphere $|\xi|_g = 1$ determined by the metric g is homologous to the sphere defined by

$$\sum_{i=1}^{2n+2} \zeta_i^2 = \xi_1^2 + \xi_2^2 + \csc^2(\eta)\xi_3^2 + \csc^2(\eta)\xi_4^2 \\ - 2\cot(\eta)\csc(\eta)\xi_3\xi_4 + \xi_5^2 + \cdots + \xi_{2n+2}^2 = 1,$$

since the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \csc^2(\eta) & -\cot(\eta)\csc(\eta) & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\cot(\eta)\csc(\eta) & \csc^2(\eta) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is positive definite. By direct calculations one can also see that in the ζ coordinates one has

$$\begin{aligned} \sigma_{\xi,\,2n+1} &= \sum_{j=1}^{2n+2} (-1)^{j-1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_{2n+2} \\ &= \sin(\eta) \sum_{j=1}^{2n+2} (-1)^{j-1} \zeta_j \, d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_{2n+2} \\ &= \sin(\eta)\ \sigma_{\zeta,\,2n+1} \end{aligned}$$

Therefore, considering Remark 3.1, we can write the Seeley–deWitt coefficient in form (3.12). □

Moreover, we need the following observation for the purpose of our description of the Seeley–deWitt coefficients as periods.

Lemma 3.5 *Only the terms with* $\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j} \in 2\mathbb{Z}$ *in (3.9) contribute nontrivially to the calculation of* α_{2n} *in* (3.12).

Proof This follows from the fact that the integral

$$\frac{1}{\sin(\eta)} \int_{|\xi|_g=1} \mathrm{tr}(\sigma_{-2n-2})\, \sigma_{\xi,\,2n+1} = \int_{Q_{2n}=1} \mathrm{tr}(\sigma_{-2n-2})\, \sigma_{\zeta,\,2n+1}$$

is independent of the variables η and ψ. Indeed, this implies that the terms in (3.9) where at least one of the integers $\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j}$ is odd cancel each other out after the integration over

$$Q_{2n} = \sum_{i=1}^{4} W_i^2 \zeta_i^2 + \sum_{i=5}^{2n+2} \zeta_i^2 = 1.$$

The terms where all the exponents are even, after the same integration, add up to an expression that is independent of the variables η and ψ. □

We introduce new coordinates, μ_1 and μ_2, defined by

$$\mu_1 = -\cos(\eta)\cos(\psi), \qquad \mu_2 = \sin(\psi),$$

and we denote by b_{-2n-2} the expression obtained from $\mathrm{tr}(\sigma_{-2n-2})$ by removing all the terms for which at least one of the $\beta_{0,1,j}, \beta_{0,2,j}, \beta_{1,1,j}, \beta_{1,2,j}$ is an odd integer. Our argument above shows that the following holds.

Corollary 3.6 *The density b_{-2n-2} is a rational expression in the variables μ_1, μ_2, $\zeta_1, \zeta_2, \ldots, \zeta_{2n+2}$ and in the affine variables $\omega_{i,j}$, $i \in \{1, 2, 3\}$, $j \in \{1, 2, \ldots, 2n\}$ determined by (3.11).*

Proof This follows directly from the previous arguments and the identities

$$\sin^2(\psi) = \mu_2^2, \qquad \cos^2(\psi) = 1 - \mu_2^2,$$
$$\sin^2(\eta) = \frac{1 - \mu_1^2 - \mu_2^2}{1 - \mu_2^2},$$
$$\cos^2(\eta) = \frac{\mu_1^2}{1 - \mu_2^2}.$$

□

For the Seeley–deWitt coefficients this then gives the following expression as a period in the algebro-geometric sense.

Theorem 3.7 *For $\omega_{i,j} \in \bar{\mathbb{Q}}$, the Seeley–deWitt coefficient $\alpha_{2n}(\omega_{i,j})$ is a period in the algebro-geometric sense, given by the integral*

$$\alpha_{2n} = \frac{1}{\pi^{n+2}} \int_{A_{2n}} \frac{b_{-2n-2}}{1 - \mu_2^2} \, d\mu_1 \wedge d\mu_2 \wedge \sigma_{\zeta,\,2n+1} \tag{3.14}$$

of an algebraic differential form

$$\frac{b_{-2n-2}}{1 - \mu_2^2} \, d\mu_1 \wedge d\mu_2 \wedge \sigma_{\zeta,\,2n+1},$$

defined on the complement in $\mathbb{A}^{2n+4}$ of the union of two hyperplanes

$$H_\pm = \{(\mu_1, \mu_2, \zeta_1, \ldots, \zeta_{2n}) \in \mathbb{A}^{2n+4} \, : \, \mu_2 = \pm 1\}$$

and the quadric defined by the vanishing of the quadratic form $Q_{W,2n}(\zeta_1, \ldots, \zeta_{2n})$*, integrated over the semi-algebraic set*

$$A_{2n} = \left\{ (\mu_1, \mu_2, \zeta_1, \zeta_2, \ldots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4}(\mathbb{R}) : 0 < \mu_1, \mu_2 < 1 \quad \textit{and} \quad \sum_{i=1}^{2n+2} \zeta_i^2 = 1 \right\}. \tag{3.15}$$

Proof This follows from the previous results, using Corollary 3.6 and the fact that

$$\sin(\eta)\, d\eta \wedge d\psi = \frac{1}{1-\mu_2^2}\, d\mu_1 \wedge d\mu_2.$$

By Proposition 3.3, the algebraic differential form $\frac{b_{-2n-2}}{1-\mu_2^2}\, d\mu_1 \wedge d\mu_2 \wedge \sigma_{\zeta,\,2n+1}$ is defined on the complement in $\mathbb{A}^{2n+4}$ of a hypersurface given by the union of two hyperplanes $H_\pm$ and the quadric $\{Q_{W,2n} = 0\}$. □

In the following section we describe the motives underlying these periods, and we show that they are mixed Tate.

4 The motives

The category of mixed Tate motives is the best understood subcategory of the more general and more mysterious category of mixed motives. Motives are a universal cohomology theory for algebraic varieties. While the case of smooth projective varieties leads to the theory of pure motives, which is better understood, modulo certain fundamental conjectures about algebraic cycles, the more general "noncompact" case of mixed motives is less well behaved in terms of categorical properties. Typically the motives that arise in physics are motives of complements of certain hypersurfaces inside an ambient algebraic variety and are therefore mixed motives. The fundamental relation between motives and periods, from the physics perspective, lies in the fact that the types of numbers that can occur as values of period integrals are strongly constrained by the nature of the motive of the algebraic variety. In particular, it is known that all periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values, [2]. While early conjectures that the periods and motives of perturbative quantum field theory would always be mixed Tate have been disproved, we show here that in this gravity model, all the coefficients of the spectral action expansion remain periods of mixed Tate motives, by explicitly computing the motive underlying the period integrals described in the previous section.

The explicit computation of the motive can be obtained in a way that is similar to the argument in the Robertson–Walker case of [11]. Due to the different choice of parameterization, the ambient space and the resulting motive are slightly different, although the main result about the mixed Tate nature of the motive is unchanged. The construction given here provides an alternative argument for the Robertson–Walker case as a particular case. We treat the variables W_i for $i = 1, \ldots, 4$ as parameters $W_i \in \mathbb{G}_m(F_i)$, where F_i are number fields. We also consider a number field F that contains the F_i.

As in [11] we adopt the following notation: we denote by $Z_{W,2n} \subset \mathbb{P}^{2n+1}$ the projective quadric determined by the quadratic form

$$Q_{W,2n}(\zeta_1, \ldots, \zeta_{2n+2}) = \sum_{i=1}^{4} W_i^2 \zeta_i^2 + \sum_{i=5}^{2n+2} \zeta_i^2, \tag{4.1}$$

for $W = (W_1, \ldots, W_4) \in \mathbb{G}_m(F)^4$,

$$Z_{W,2n} = \{(\zeta_1 : \ldots : \zeta_{2n+2}) \in \mathbb{P}^{2n+1} \,:\, Q_{W,2n}(\zeta_1, \ldots, \zeta_{2n+2}) = 0\}.$$

We also denote by $C^2 Z_{W,2n}$ the projective cone of $Z_{W,2n}$ in $\mathbb{P}^{2n+3}$ and we denote by $\widehat{Z}_{W,2n}$ the affine cone in $\mathbb{A}^{2n+2}$ and by $\widehat{C^2 Z}_{W,2n}$ the affine cone of $C^2 Z_{W,2n}$ in $\mathbb{A}^{2n+4}$.

We are interested in the mixed motive

$$\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n}), \Sigma), \tag{4.2}$$

where $H_\pm$ are the hyperplanes

$$H_\pm = \{(\mu_1, \mu_2, \zeta_1, \ldots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4} \,:\, \mu_2 = \pm 1\} \tag{4.3}$$

and Σ is the divisor in $\mathbb{A}^{2n+4}$ given by

$$\Sigma = \cup_{i=1}^{2} \cup_{j=0}^{1} H_{i,j},$$

where $H_{i,j}$ are the hyperplanes

$$H_{i,j} = \{(\mu_1, \mu_2, \zeta_1, \ldots, \zeta_{2n+2}) \in \mathbb{A}^{2n+4} \,:\, \mu_i = j\}.$$

We first give an explicit computation of the class

$$[\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n})] \tag{4.4}$$

in the Grothendieck ring of varieties $K_0(\mathcal{V}_F)$ with F an extension of the F_i that also contains the number field $\mathbb{Q}(\sqrt{-1})$, and then we prove that motive (4.2) is mixed Tate (as a motive over F).

4.1 The quadratic form and field extensions

Let F be a number field that contains the fields F_i, for $i = 1, \dots, 4$ and $\mathbb{Q}(\sqrt{-1})$. Over F consider the change of variables

$$\begin{array}{ll} X_1 = W_1\zeta_1 + iW_2\zeta_2, & Y_1 = W_1\zeta_1 - iW_2\zeta_2, \\ X_2 = i(W_3\zeta_3 + iW_4\zeta_4), & Y_2 = i(W_3\zeta_3 - iW_4\zeta_4). \end{array} \tag{4.5}$$

In these variables the quadratic form $Q_{W,2}$ becomes the quadratic form

$$X_1Y_1 - X_2Y_2,$$

hence the projective quadric $Z_{W,2} \subset \mathbb{P}^3$ is the Segre quadric

$$Z_{W,2} = \{X_1Y_1 - X_2Y_2 = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

Moreover, over the same field F the further changes of coordinates

$$X_n = \zeta_{2n-1} + i\zeta_{2n}, \quad Y_n = \zeta_{2n-1} - i\zeta_{2n} \tag{4.6}$$

transform the quadratic form $Q_{W,2n}$ into the form

$$Q_{W,2n-2}(\zeta_1, \dots, \zeta_{2n}) + X_nY_n. \tag{4.7}$$

4.2 The Grothendieck class

The Grothendieck ring of varieties $K_0(\mathcal{V}_F)$ is generated by isomorphism classes $[X]$ of varieties over F with the inclusion–exclusion relation $[X] = [Y] + [X \smallsetminus Y]$ for closed subvarieties $Y \hookrightarrow X$ and the product relation $[X] \cdot [Y] = [X \times Y]$. In order to compute Grothendieck class (4.4), we use the following facts, which are a variant of Lemma 4.1 of [11].

Lemma 4.1 *Let $Z \subset \mathbb{P}^{2n+1}$ is a projective hypersurface and let $C^2Z \subset \mathbb{P}^{2n+3}$, $\hat{Z} \subset \mathbb{A}^{2n+2}$ and $\widehat{C^2Z} \subset \mathbb{A}^{2n+4}$ be the projective and affine cones as above. Also let $H_\pm$ be two hyperplanes in $\mathbb{A}^{2n+4}$ with $H_+ \cap H_- = \emptyset$ and with intersections $H_\pm \cap \widehat{C^2Z}$ given by sections of the cone. Then the Grothendieck classes satisfy*

- $[\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2Z}] = \mathbb{L}^{2n+4} - \mathbb{L}^3[Z] + \mathbb{L}^2([Z] - 1)$,
- $[\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2Z} \cup H_+ \cup H_-)] = \mathbb{L}^{2n+4} - 2\mathbb{L}^{2n+3} - \mathbb{L}^3[Z] + 3\mathbb{L}^2[Z] - 2\mathbb{L}[Z] - \mathbb{L}^2 + 2\mathbb{L}$.

Proof Let $\mathbb{L} = [\mathbb{A}^1]$ be the Lefschetz motive, the Grothendieck class of the affine line. We have $[\mathbb{A}^{2n+2} \smallsetminus \hat{Z}] = (\mathbb{L} - 1)[\mathbb{P}^{2n+1} \smallsetminus Z]$ since $[\hat{Z}] = (\mathbb{L} - 1)[Z] + 1$. Moreover, we have $[CZ] = \mathbb{L}[Z] + 1$, since the projective cone is the union of a copy of Z and a copy of the affine cone $\hat{Z}$. Similarly, we have $[C^2Z] = \mathbb{L}[CZ] + 1 = \mathbb{L}^2[Z] + \mathbb{L} + 1$. We then have

$$[\mathbb{A}^{2n+4} \setminus \widehat{C^2Z}] = (\mathbb{L}-1)[\mathbb{P}^{2n+3} \setminus C^2Z] = \mathbb{L}^{2n+4} - 1 - (\mathbb{L}-1)[C^2Z]$$
$$= \mathbb{L}^{2n+4} - 1 - (\mathbb{L}-1)(\mathbb{L}^2[Z] + \mathbb{L} + 1) = \mathbb{L}^{2n+4} - \mathbb{L}^3[Z] + \mathbb{L}^2([Z]-1).$$

By inclusion–exclusion we have

$$[\widehat{C^2Z} \cup H_+ \cup H_-] = [\widehat{C^2Z}] + [H_- \cup H_+] - [\widehat{C^2Z} \cap (H_+ \cup H_-)]$$
$$= [\widehat{C^2Z}] + 2\mathbb{L}^{2n+3} - 2[\widehat{CZ}],$$

where $[\widehat{C^2Z}] = (\mathbb{L}-1)[C^2Z] + 1 = \mathbb{L}^3[Z] + \mathbb{L}^2 - \mathbb{L}^2[Z] = \mathbb{L}^3[Z] - \mathbb{L}^2([Z]-1)$ and $[\widehat{CZ}] = (\mathbb{L}-1)[CZ] + 1 = \mathbb{L}^2[Z] - \mathbb{L}([Z]-1)$, so that we obtain

$$[\widehat{C^2Z} \cup H_+ \cup H_-] = \mathbb{L}^3[Z] - \mathbb{L}^2([Z]-1) + 2\mathbb{L}^{2n+3} - 2(\mathbb{L}^2[Z] - \mathbb{L}([Z]-1))$$
$$= 2\mathbb{L}^{2n+3} + \mathbb{L}^3[Z] - 3\mathbb{L}^2[Z] + 2\mathbb{L}[Z] + \mathbb{L}^2 - 2\mathbb{L}.$$

□

Proposition 4.2 *Let $Q_{W,2n}$ be quadratic form (4.1) and $Z_{W,2n} \subset \mathbb{P}^{2n+1}$ the projective quadric defined by the vanishing of $Q_{W,2n}$. Let $C_{2n} = [\mathbb{A}^{2n+2} \setminus \hat{Z}_{W,2n}]$ be the Grothendieck class in $K_0(\mathcal{V}_F)$ of the affine hypersurface complement. This is given by*

$$C_{2n} = \mathbb{L}^{2n+2} - \mathbb{L}^{2n+1} - \mathbb{L}^{n+1} + \mathbb{L}^n. \tag{4.8}$$

Similarly, we have $[Z_{W,2n}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1} + 2\mathbb{L}^n + \mathbb{L}^{n+1} + \cdots + \mathbb{L}^{2n}$ and

$$[\mathbb{A}^{2n+4} \setminus \widehat{C^2Z}_{W,2n}] = \mathbb{L}^{2n+4} - \mathbb{L}^{2n+3} - \mathbb{L}^{n+3} + \mathbb{L}^{n+2}. \tag{4.9}$$

$$[\mathbb{A}^{2n+4} \setminus (\widehat{C^2Z}_{W,2n} \cup H_+ \cup H_-)] = \mathbb{L}^{2n+4} - 3\mathbb{L}^{2n+3} + 2\mathbb{L}^{2n+2} - \mathbb{L}^{n+3} + 3\mathbb{L}^{n+2} - 2\mathbb{L}^{n+1}. \tag{4.10}$$

for $H_\pm = \{\mu_2 = \pm 1\} \subset \mathbb{A}^{2n+4}$.

Proof We first show that the classes C_{2n} satisfy the recursive formula

$$C_{2n} = \mathbb{L}^{2n+2} - 2\mathbb{L}^{2n+1} + \mathbb{L}^{2n} + \mathbb{L}C_{2n-2}. \tag{4.11}$$

To see this, consider the condition that $Q_{W,2n} \neq 0$. By (4.7), using the change of variables (4.6) over F, this is equivalent to

$$Q_{W,2n-2}(\zeta_1, \ldots \zeta_{2n}) + X_n Y_n \neq 0.$$

Suppose $X_n = 0$. Then $Y_n \in \mathbb{A}^1$ and $Q_{W,2n-2}(\zeta_1, \ldots \zeta_{2n}) \neq 0$. Thus, this case contributes a term $\mathbb{L} \cdot C_{2n-2}$ to the class C_{2n}. The case $X_n \neq 0$ gives

$$Y_n \neq \frac{Q_{W,2n-2}(\zeta_1, \ldots \zeta_{2n})}{X_n},$$

which gives $(\zeta_1, \ldots \zeta_{2n}) \in \mathbb{A}^{2n}$ and $Y_n \in \mathbb{G}_m$ with $X_n \in \mathbb{G}_m$. Thus, this case contributes a term $[\mathbb{G}_m]^2 \mathbb{L}^{2n} = \mathbb{L}^{2n}(\mathbb{L} - 1)^2$. This gives $C_{2n} = \mathbb{L}^{2n+2} - 2\mathbb{L}^{2n+1} + \mathbb{L}^{2n} + \mathbb{L} \cdot C_{2n-2}$. We can then verify (4.8) by induction. When $n = 1$, we know from Sect. 4.1 that the change of variables (4.5) over F transforms the quadric $Q_{W,2}$ into the quadric $X_1 Y_1 - X_2 Y_2$, hence $[Z_{W,2}] = [\mathbb{P}^1 \times \mathbb{P}^1] = \mathbb{L}^2 + 2\mathbb{L} + 1$ in $K_0(\mathcal{V}_F)$. Thus we have $[\hat{Z}_{W,2}] = (\mathbb{L} - 1)[Z_{W,2}] + 1 = (\mathbb{L} - 1)(\mathbb{L}^2 + 2\mathbb{L} + 1) + 1 = \mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} - \mathbb{L}^2 - 2\mathbb{L} - 1 + 1 = \mathbb{L}^3 + \mathbb{L}^2 - \mathbb{L}$, hence $C_2 = \mathbb{L}^4 - \mathbb{L}^3 - \mathbb{L}^2 + \mathbb{L}$. Then suppose that $C_{2n-2} = \mathbb{L}^{2n} - \mathbb{L}^{2n-1} - \mathbb{L}^n + \mathbb{L}^{n-1}$. We obtain $C_{2n} = \mathbb{L}^{2n+2} - 2\mathbb{L}^{2n+1} + \mathbb{L}^{2n} + \mathbb{L}(\mathbb{L}^{2n} - \mathbb{L}^{2n-1} - \mathbb{L}^n + \mathbb{L}^{n-1}) = \mathbb{L}^{2n+2} - \mathbb{L}^{2n+1} - \mathbb{L}^{n+1} + \mathbb{L}^n$. We also have $[\hat{Z}_{W,2n}] = \mathbb{L}^{2n+1} + \mathbb{L}^{n+1} - \mathbb{L}^n$ and $[Z_{W,2n}] = ([\hat{Z}_{W,2n}] - 1)(\mathbb{L} - 1)^{-1} = (\mathbb{L}^{2n+1} + \mathbb{L}^{n+1} - \mathbb{L}^n - 1)(\mathbb{L} - 1)^{-1} = 1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n$, hence

$$\begin{aligned}
[\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n}] &= \mathbb{L}^{2n+4} - \mathbb{L}^3 [Z_{W,2n}] + \mathbb{L}^2([Z_{W,2n}] - 1) \\
&= \mathbb{L}^{2n+4} - \mathbb{L}^3(1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n) \\
&\quad + \mathbb{L}^2(1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n) - \mathbb{L}^2 \\
&= \mathbb{L}^{2n+4} - \mathbb{L}^{2n+3} - \mathbb{L}^{n+3} + \mathbb{L}^{n+2},
\end{aligned}$$

as the other terms cancel in a telescopic sum. Similarly, we have

$$\begin{aligned}
&[\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-)] = \mathbb{L}^{2n+4} - 2\mathbb{L}^{2n+3} - \mathbb{L}^3[Z_{W,2n}] + 3\mathbb{L}^2[Z_{W,2n}] \\
&\quad - 2\mathbb{L}[Z_{W,2n}] - \mathbb{L}^2 + 2\mathbb{L} = \mathbb{L}^{2n+4} - 2\mathbb{L}^{2n+3} - \mathbb{L}^3(1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n) \\
&\quad + 3\mathbb{L}^2(1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n) - 2\mathbb{L}(1 + \mathbb{L} + \cdots + \mathbb{L}^{2n} + \mathbb{L}^n) - \mathbb{L}^2 + 2\mathbb{L} \\
&= \mathbb{L}^{2n+4} - 3\mathbb{L}^{2n+3} + 2\mathbb{L}^{2n+2} - \mathbb{L}^{n+3} + 3\mathbb{L}^{n+2} - 2\mathbb{L}^{n+1}.
\end{aligned}$$

□

4.3 The mixed motive

The result of Proposition 4.2 shows that the Grothendieck classes in $K_0(\mathcal{V}_F)$ of the complements $\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n}$ and $\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-)$ are in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_F)$. We now consider mixed motive (4.2), as an element in the Voevodsky triangulated category of mixed motives, [30], and we show that it is in the triangulated subcategory of mixed Tate motives. The argument is analogous to Theorem 4.3 and Proposition 4.5 of [11]. We present it here explicitly for completeness. Here, as above, we consider a number field F that contains the number fields F_i with $W_i \in \mathbb{G}_m(F_i)$ and also contains $\mathbb{Q}(\sqrt{-1})$.

Theorem 4.3 *The mixed motive $\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_- \cup \widehat{C^2 Z}_{W,2n}), \Sigma)$ over the number field F is a mixed Tate motive.*

Proof Using the change of variables (4.6) we see that over the field extension F the quadratic form becomes isotropic, namely $Q_{W,2n}|_F = (n+1)\mathbb{H}$, where $\mathbb{H} = \langle 1, -1\rangle$ is the hyperbolic quadratic form. This implies that, over F, the motive $\mathfrak{m}(Z_{W,2n})$ is given by (see [28])

$$\mathfrak{m}(Z_{W,2n}) = \mathbb{Z}(n)[2n] \oplus \mathbb{Z}(n)[2n] \oplus \bigoplus_{k=0,\dots,n-1,n+1,\dots,2n} \mathbb{Z}(k)[2k].$$

This motivic decomposition of the motive corresponds to the expression $[Z_{W,2n}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1} + 2\mathbb{L}^n + \mathbb{L}^{n+1} + \cdots + \mathbb{L}^{2n}$ for the Grothendieck class. The Gysin distinguished triangle in the Voevodsky category gives

$$\mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n}) \to \mathfrak{m}(\mathbb{P}^{2n+1}) \to \mathfrak{m}(Z_{W,2n})(1)[2] \to \mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n})[1].$$

Since two of the three terms, $\mathfrak{m}(\mathbb{P}^{2n+1})$ and $\mathfrak{m}(Z_{W,2n})(1)[2]$, are mixed Tate, the third term $\mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n})$ is also mixed Tate. Note then that, when taking a projective cone, the map $\mathbb{P}^{2n+2} \smallsetminus CZ_{W,2n} \to \mathbb{P}^{2n+1} \smallsetminus Z_{W,2n}$ is an $\mathbb{A}^1$-fibration, and so is the map $\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n} \to \mathbb{P}^{2n+2} \smallsetminus CZ_{W,2n}$. By homotopy invariance of the Voevodsky motive, we then have $\mathfrak{m}(\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n}) \simeq \mathfrak{m}(\mathbb{P}^{2n+1} \smallsetminus Z_{W,2n})$. Thus, the motive $\mathfrak{m}(\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n})$ is also mixed Tate. The relation between the motive $\mathfrak{m}(\mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n})$ and the motive $\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n})$ is obtained by considering the $\mathbb{G}_m$-bundle $\mathcal{T} = \mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n} \to \mathbb{P}^{2n+3} \smallsetminus C^2 Z_{W,2n}$ and the associated $\mathbb{P}^1$-bundle $\mathcal{P}$ and the Gysin distinguished triangle of [30], p. 197,

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1].$$

The motive of a $\mathbb{P}^1$-bundle over a base X satisfies $\mathfrak{m}(\mathcal{P}) = \mathfrak{m}(X) \oplus \mathfrak{m}(X)(1)[2]$; hence, it is mixed Tate if $\mathfrak{m}(X)$ is mixed Tate. The motive $\mathfrak{m}(\mathcal{P} \smallsetminus \mathcal{T})$ is mixed Tate because it consists of two copies of X. Thus, in the above triangle both $\mathfrak{m}(\mathcal{P})$ and $\mathfrak{m}(\mathcal{P} \smallsetminus \mathcal{T})$ are mixed Tate; hence, the third term $\mathfrak{m}(\mathcal{T}) = \mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n})$ is also mixed Tate. Next we show that the motive $\mathfrak{m}(\mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-))$ is mixed Tate as well. We use the Mayer–Vietoris distinguished triangle in the Voevodsky category

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1],$$

applied to the open sets $U = \mathbb{A}^{2n+4} \smallsetminus \widehat{C^2 Z}_{W,2n}$ and $V = \mathbb{A}^{2n+4} \smallsetminus (H_+ \cup H_-)$, with $U \cup V = \mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-))$ and $U \cap V = \mathbb{A}^{2n+4} \smallsetminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup$

H_-). We want to show that $\mathfrak{m}(U \cap V)$ is mixed Tate. By the Mayer–Vietoris triangle it suffices to show that $\mathfrak{m}(U)$, $\mathfrak{m}(V)$, and $\mathfrak{m}(U \cup V)$ are all mixed Tate. We know that $\mathfrak{m}(U)$ is mixed Tate by the previous argument. To see that $\mathfrak{m}(V)$ is mixed Tate observe that $\mathfrak{m}(H_+ \cup H_-)$ certainly is; hence, the Gysin triangle ensures that $\mathfrak{m}(V)$ is also mixed Tate. In the case of $\mathfrak{m}(U \cup V)$, the intersection $\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-)$ consists of two sections of the cone, isomorphic to $\widehat{CZ}_{W,2n}$, hence $\mathfrak{m}(\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-)) = \mathfrak{m}(\widehat{CZ}_{W,2n}) \oplus \mathfrak{m}(\widehat{CZ}_{W,2n})$. The motive $\mathfrak{m}(\widehat{CZ}_{W,2n})$ is mixed Tate because the motive of the complement is by the previous argument about homotopy invariants and the Gysin triangle. Thus, the motive of the complement $\mathfrak{m}(\mathbb{A}^{2n+4} \setminus (\widehat{C^2 Z}_{W,2n} \cap (H_+ \cup H_-))$ is also mixed Tate, again by an application of the Gysin triangle. Thus, by Mayer–Vietoris we have obtained that $\mathfrak{m}(\mathbb{A}^{2n+4} \setminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-))$ is mixed Tate. Finally, motive (4.2) also fits in a distinguished triangle where two of the terms, $\mathfrak{m}(A^{2n+4} \setminus (\widehat{C^2 Z}_{W,2n} \cup H_+ \cup H_-))$ and $\mathfrak{m}(\Sigma)$, are mixed Tate; hence, it is also mixed Tate. □

Remark 4.4 Assuming for simplicity that $W_i \in \mathbb{G}_m(\mathbb{Q})$, the motive $\mathfrak{m}(Z_{W,2n})$ over $\mathbb{Q}$, where the quadratic form $Q_{W,2n}$ is not isotropic, can be expressed in terms of "forms of Tate motives," which become Tate motives after passing to a field extension. These are the Rost motives of quadrics, see [25,27,28], and §4.6 of [11].

Remark 4.5 In [8] we proved that, for Bianchi IX metrics that are gravitational instantons (Einstein and self-dual), the heat-kernel coefficients are vector-valued modular forms. Thus, we see two different arithmetic structures associated with these heat-kernel coefficients: as we have shown here, if one fixes an algebraic value of the anisotropy coefficients w_i (hence of the coefficients W_i), then the corresponding Seeley–deWitt coefficients are periods of mixed Tate motives; on the other hand, if one considers the w_i and an overall conformal factor F as functions of the cosmological time μ and the two parameters (p, q) determining the family of solutions of the gravitational instanton equations, then the Seeley–deWitt coefficients are vector-valued (meromorphic) modular forms in the variable $i\mu$ in the upper half plane.

Finally, it is worth pointing out that the spectral action is a model of *Euclidean gravity*, since its construction and properties rely essentially on the analytic properties of the spectrum of the Dirac operators on compact Riemannian manifolds. While some results exist that relate the spectral action formalism to Lorentzian geometry, there is no good general framework to translate results about the spectral action to the indefinite signature case. It is often the case that the local expressions involved in the asymptotic expansion of the spectral action continue to make sense when Wick rotated to Euclidean signature. This fact is used frequently, for example, in cosmological applications of the spectral action (see [21]). However, while there are many interesting examples of Lorentzian nonisotropic spacetimes (Kasner metrics, mixmaster universe models), the Bianchi IX gravitational instantons that we consider in this paper are also specific to Euclidean signature and especially relevant in the context of Euclidean quantum gravity and quantum cosmology. It is possible that certain classes of Lorentzian spacetimes may also have interesting arithmetic structures related to the heat-kernel expansion of specific geometric differential operators.

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