MODULAR NORI MOTIVES

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ABSTRACT. In a previous article [CMMar20], we developed the pioneering Grothendieck approach to the problem of description of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ based upon *dessins d'enfant.* Namely, we replaced in it dessins d'enfant by graphs encoding combinatorics of strata of modular spaces of genus zero $\overline{M}_{0,n}$, and applied this new category to the study of quantum statistic properties of the absolute Galois group.

In this short paper, we enrich and further develop this approach by including in this picture the Nori motives of the strata of modular spaces following [MaMar19-1].

CONTENTS

0. Introduction and summary

In the Introduction to [CMMar20], we reminded that Grothendieck's approach to the study of the profinite completion of "absolute Galois group" $G_{\mathbf{Q}}$ of the field of all algebraic numbers started with observation that for any integral scheme X, $G_{\mathbf{Q}}$ acts by outer automorphisms upon étale fundamental group of $X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ via exact sequence

$$1 \to \pi_1(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}) \to \pi_1(X) \to G_{\mathbf{Q}} \to 1 \qquad (0.1)$$

In [CMMar20], we replaced unramified covers of $X = \mathbf{P}^1 \setminus \{0, 1, \infty\}$ producing the classical dessins d'enfant, by the family of moduli spaces $\overline{M}_{0,S}$ of stable genus zero algebraic curves with labelled points, and their forms in the sense of [Se13]. We have shown that the role of dessins d'enfant can be played by *dual combinatorial graphs* τ of such curves each of which determines the so called (closed) stratum $\overline{M}_{0,\tau}$ in a respective moduli space.

Here we enrich the combinatorics of strata by passing from it to the *Nori motives of strata* as in [MaMar19-1]. Our basic reference for Nori motivic theory is [HuM-St17].

One of our motivations in [CMMar20] was the desire to avoid model structures and homotopy equivalences used for comparison of Grothendieck–Teichmüller constructions with modular ones in the works by P. de Brito, G. Horel and M. Robertson, and most recently in [DoShVaiVal20]. Our enrichment by Nori motives of strata does not avoid model structures either, but arguably, uses them in a more natural and condensed context. Concretely, in this context the diagram (0.1) is replaced by a subdiagram (generated by modular dessins) of the "motivic fundamental groups"

$$1 \to G_{\mathrm{mot}}(\overline{\mathbf{Q}}, \mathbf{Q}) \to G_{\mathrm{mot}}(\mathbf{Q}, \mathbf{Q}) \to G_{\mathbf{Q}} \to 1, \tag{0.2}$$

see [HuM–St17], Theorem 9.1.16, and further explanations in this article.

Here is a sketch of the contents of two parts of it.

The first part (Section 1) is a technical survey of general Nori constructions based upon Chapter 9 of [HuM–St17].

In the second part (Section 2), we describe our "modular Nori motives" as motives of strata of genus zero moduli spaces $\overline{M}_{0,S}$. This is done in terms of combinatorics of "modular dessins", encoding stable degenerations of $\mathbf{P}^1$ with ≥ 4 marked points. The goal of this study consists in the enrichment of quantum statistical machinery properties of the Grothendieck–Teichmüller group that were already introduced and described in [CMMar20].

In order to appreciate Nori's contribution to algebraic geometry , we recommend to the reader to look at the last pages of P. Cartier's essay [Ca14], clarifying the initial Grothendieck's vision of cohomology theories in algebraic geometry during his last years at IHES.

Cartier expressed this vision by the following words:

"the various known cohomological theories [...] are what we see, and it is necessary to go back to the source and build the lighthouse which will unify the representation of the entire" [...] "rocky coastline at night".

This metaphor is embodied in the Nori's construction of motives as *universal* cohomology: see Sec. 1.5 and further on.

1. Survey of Nori motives

1.1. **Categories of diagrams and graphs.** In [CMMar20], a version of Grothendieck's *dessins* in modular environment was defined as objects of a category of combinatorial graphs. In the definition of Nori motives in [HuM–St17], the key role is played by a category of diagrams. Graphs and diagrams are close but not identical objects, and we will start with a brief description of their interconnections.

a) A *diagram* D is a family (in a fixed small universe), consisting of two disjoint sets $V(D)$ (vertices), $E(D)$ (edges), and a map $\partial : E(D) \to V(D) \times V(D)$, $\partial(e) = (\partial_{out}(e), \partial_{in}(e))$ (orientation of edges). An oriented edge is sometimes called *an arrow.*

Morphism of diagrams $D_1 \to D_2$ consists of two maps $V(D_1) \to V(D_2)$, $E(D_1) \to E(D_2)$, compatible with orientations.

A diagram *with identities* is a diagram D in which for every vertex v, exactly one oriented edge from v to v is given and called the identity edge id_v. Morphism of diagrams with identities must map identities to identities.

It is easy to check, that a subset of diagrams and a subset of their pairwise morphisms, stable with respect to pairwise composition of morphisms, and containing all identity morphisms, forms a category, because composition of morphisms of diagrams is associative.

b) Similarly, a *combinatorial graph* τ is defined as a family of sets and maps (F_τ, V_τ), (∂_τ, j_τ).

Elements of F_τ are called *flags* of τ, elements of V_τ are called *vertices* of τ.

Map $\partial_\tau : F_\tau \to V_\tau$ is called the *boundary map.* Map $j_\tau : F_\tau \to F_\tau$ is called the *structure involution.* It must satisfy the condition $j_\tau^2 = id$.

Two–element orbits of j_τ form the set E_τ of *edges* of τ. Elements of one such orbit are sometimes called "halves" of the respective edge, and two points, – boundaries of a member of this orbit, – the boundary of the respective edge itself.

One–element orbits of j_τ are called *tails*, or *leaves* (we will use both words as synonymous). A graph τ with one vertex and no edges is called *corolla.*

The detailed definition of morphisms of combinatorial graphs is neither short, nor self–evident: see details in [BoMa07].

Comparing the two definitions (of a diagram and of a graph), one sees that *each diagram D is in fact an oriented graph τ*, that is a graph, *which has no tails, and in which orientation of each edge is added as a structure element.*

More precisely, $V_\tau := V(D)$, $F_\tau := F(D)_{out} \sqcup F(D)_{in}$ where

$$F(D)_{out} := \{(v, \partial_{out}(e)) \,|\, e \in E(D)\}, \quad F(D)_{in} := \{(v, \partial_{in}(e)) \,|\, e \in E(D)\},$$

and j_τ interchanges $(v, \partial_{out}(e))$ with $(v, \partial_{in}(e))$.

Consider now a category of diagrams with identities. One easily sees that when we replace the diagrams by their combinatorial graphs, morphisms of diagrams become the morphisms of the respective graphs, so this replacement becomes a functor.

Vice versa, each category $\mathcal{C}$ defines a diagram with identities $D(\mathcal{C})$ for which

$$V(D(\mathcal{C})) := \mathrm{Ob}\,\mathcal{C}, \quad E(D(\mathcal{C})) := \mathrm{Hom}_{\mathcal{C}},$$

and $\partial(f : X \to Y) := (X, Y)$.

Given a diagram D and a category $\mathcal{H}$, any morphism of diagrams $D \to D(\mathcal{H})$ is called *a representation* of D. Of course, representations (perhaps, satisfying additional compatibility conditions) themselves are objects of a category/ vertices of its diagram etc. This is the universe where the construction of Nori motives is developing.

1.2. **Linear representations of diagrams.** Start with the following data, that intuitively will encode a category of geometric objects whose (co)homology theories we want to construct:

a) a diagram D;

b) a noetherian commutative ring with unit R and the category of finitely generated R–modules R–Mod;

c) a representation T of D in R–Mod.

Let $End(T)$ be defined as the ring

$$End(T) := \{(\varphi_v)_{v\in V(D)} \,|\, \varphi_v \in End_R(T(v)) \textit{ such that}$$

$$\varphi_{\partial_{in}(e)} \circ T(e) = T(e) \circ \varphi_{\partial_{out}(e)}, \ \forall e \in E(D)\}.$$

An inclusion of diagrams $D_1 \subset D_2$ such that $T_1 = T_2|_{D_1}$ determines a homomorphism $End(T_2) \to End(T_1)$, by projecting the product $\prod_{v \in V(D_2)} End_R(T_2(v))$ onto the product $\prod_{v \in V(D_1)} End_R(T_1(v))$.

Now produce from the data above the category $C(D,T)$ defined in the following way:

d1) If D is finite, then $C(D,T)$ is the category $End(T)$-Mod of finitely generated R–modules equipped with an R–linear action of $End(T)$.

d2) If D is infinite, first consider all of its finite subdiagrams F.

For each F construct $C(F, T|_F)$ as in d1). Then apply the following limiting procedure. *Objects of* $C(D,T)$ will be all objects of the categories $C(F, T|_F)$. If $F \subset F'$, then each object X_F of $C(F, T|_F)$ gives an object of $X_{F'}$ of $C(F', T|_{F'})$, via the map from $End(T_F)$-Mod to $End(T_{F'})$-Mod determined by the morphism $End(T_{F'}) \to End(T_F)$ as above. *Morphisms from* X *to* Y in $C(D,T)$ will be defined as colimits over F of morphisms from X_F to Y_F with respect to these extensions.

The result is called *the diagram category* $C(D,T)$. It is an R–linear abelian category which is endowed with R–linear faithful exact forgetful functor

$f_T : C(D,T) \to R$-Mod .

This diagram category has the following universal property.

Given any R–linear abelian category A with a representation $F : D \to A$ and R–linear faithful exact functor $f : A \to R$-Mod with $T = f \circ F$, it factorises through a faithful exact functor $L(F) : C(D,T) \to A$ compatibly with decomposition

$$T = f_T \circ \tilde{T},\ \tilde{T} : D \to C(D,T).$$

For proofs and more details, see [HuM-St17], pp. 140-144.

1.3. **Multiplicativity.** Here we sketch the basic constructions introducing *multiplicative structures* on categories of Nori motives, following [HuM-St17]. More detailed discussion of multiplicative structures (called *tensor structures* there) the reader can find in [KashScha06], Chapter 4 and further on.

Consider two diagrams with identities D_1, D_2. We define the diagram $D_1 \times D_2$ in the following way. Its vertices are ordered pairs of vertices of D_1, D_2

$$V(D_1 \times D_2) := V(D_1) \times V(D_2).$$

Its edges are ordered pairs of edges of the form (e, id) or (f, id), with obvious boundary map ∂.

The standard list of axioms can be found in [HuM-St17], 8.1.3. What is called "graded diagram" and "graded multiplication" there, we would prefer to call "supergrading". Anyway, in our treatment of modular Nori motives below, these restrictions may be omitted, essentially, because cohomology of modular motives vanishes in odd dimensions.

1.4. **Rigidity.** The first important result is that the multiplicative structure of the category of diagrams induces the multiplicative structure of the category of its representations $C(D,T)$. If R is a field or a Dedekind domain, then $C(D,T)$ is equivalent to the category of comodules of finite type over a coalgebra $A(D,T)$: see [HuM-St17], Theorem 7.1.12. Moreover, $A(D,T)$ carries a natural structure of commutative bialgebra, with unit and counit.

The scheme $M := \mathrm{Spec}(A(D,T))$ is faithfully flat unital monoid scheme over $\mathrm{Spec}R$. Notice that here the coefficient ring of cohomology theory k enters the game, because its spectrum becomes the final object of the relevant geometric realisations of (co)homology theories.

The notion of rigidity is the last important property of relevant multiplicative categories of diagrams C, needed for the construction of the exact sequence (0.2).

Briefly, it requires the existence of a dualisation functor $V \mapsto V^{\vee}$ related to the multiplication (here written as $\otimes$) by natural identifications

$$\mathrm{Hom}(X \otimes V, Y) \cong \mathrm{Hom}(X, V^{\vee} \otimes Y)$$

and $V^{\vee\vee} \cong V$.

For further details and the definition of the motivic groups G_{mot} in (0.2), see Section 9.5 of [HuM-St17].

1.5. **Nori geometric diagrams and Nori motives.** Consider a geometric category $\mathcal{C}$ of spaces/varieties/schemes, in which one can define morphisms of closed embeddings $Y \hookrightarrow X$ (or $Y \subset X$) and morphisms of complements to closed embeddings $X \setminus Y \to X$. We can then define the Nori diagram of *effective pairs* $D(\mathcal{C})$ in the following way (see [HuM-S17], pp. 207–208).

a). One vertex of $D(\mathcal{C})$ is a triple (X, Y, i) where $Y \hookrightarrow X$ is a closed embedding, and i is an integer.

b). Besides obvious identities, there are edges of two types.

b1). Let (X,Y) and (X',Y') be two pairs of closed embeddings. Every morphism $f : X \to X'$ such that $f(Y) \subset Y'$ produces functoriality edges f^* (or rather (f^*,i)) going from (X',Y',i) to (X,Y,i).

b2). Let $(Z \subset Y \subset X)$ be a stair of closed embeddings. Then it defines coboundary edge ∂ from (Y,Z,i) to $(X,Y,i+1)$.

We can now pass to (co)homological representations of Nori geometric diagrams. If we start not just from the initial category of spaces $\mathcal{C}$, but rather from a pair $(\mathcal{C},H)$ where H is a cohomology theory, then assuming reasonable properties of this pair, we can define the respective representation T_H of $D(\mathcal{C})$ that we will call a *(co)homological representation of* $D(\mathcal{C})$.

For a survey of such pairs $(\mathcal{C},H)$ that were studied in the context of Grothendieck's motives, see [HuM–St17], pp. 131–133. The relevant cohomology theories include, in particular, singular cohomology, and algebraic and holomorphic de Rham cohomologies.

Below we will consider the basic example of cohomological representations of Nori diagrams that leads to Nori motives.

1.6. Effective Nori motives. ([HuM–St17], pp. 207–208.) Take as the starting object a category $\mathcal{C}$ of varieties X defined over a subfield $k \subset \mathbf{C}$.

We can then define the Nori diagram $D(\mathcal{C})$ as above. This diagram will be denoted $Pairs^{eff}$ from now on.

1.7. Other main categories of Nori motives. We introduce categories $\mathcal{MM}$ of mixed Nori motives, denoted also $\mathcal{MM}(k)$ when we want to stress the base field/ring. In [HuM-St17], Definition 9.1.3, they are denoted $\mathcal{MM}_{\text{Nori}}$, $\mathcal{MM}_{\text{Nori}}(k)$, etc.

Namely, by definition,$\mathcal{MM}^{\text{eff}}_{\text{Nori}}(k)$ is $C(Pairs^{\text{eff}},H^*)$, and $\mathcal{MM}_{\text{Nori}}(k)$ is the localisation of $\mathcal{MM}^{\text{eff}}_{\text{Nori}}(k)$ by the Lefschetz motive.

1.8. Theorem. *If k is a subfield of* $\mathbf{C}$*, $\mathcal{MM}(k)$ is a rigid tensor category equivalent to the category of representations of a faithfully flat proalgebraic group scheme $G_{\text{mot}}(k,R)$.*

2. Modular Nori motives and quantum statistical mechanics of the absolute Galois group

In this Section we will use the relevant definitions and notations from Sec. 2 of [CMMar20]. We will collect here only those properties of canonical stratifications of moduli spaces $\overline{M}_{0,S}$ that are essential for the understanding of their Nori motives.

The strata we have in mind are naturally numbered by *combinatorial graphs* encoding stable curves of genus zero with a finite subset of marked/labelled nonsingular points. We will start working over the field of algebraic numbers $\overline{\mathbf{Q}}$, or more generally, over any algebraically closed subfield of $\mathbf{C}$.

2.1. Moduli spaces and their canonical stratifications. For any a finite set S of cardinality $n+1 \geq 4$, the stable genus zero curves with $n+1$ points labelled by S are parametrised by points of the smooth projective irreducible manifold $\overline{M}_{0,S}$ of dimension $n-2$.

The subspace of points corresponding to only *irreducible* curves is an open Zariski dense submanifold $M_{0,S} \subset \overline{M}_{0,S}$. It parametrizes curves whose graph is a corolla with S tails.

More generally, for any a stable connected tree τ with the set of tails labelled by S, all stable genus zero modular curves with graph τ and their further specialisations/degenerations are parametrised by the Zariski closed smooth projective manifold $\overline{M}_{0,\tau} \subset \overline{M}_{0,S}$.

Those curves whose graph is *exactly* τ are parametrised by the Zariski open dense submanifold $M_{0,\tau} \subset \overline{M}_{0,\tau}$.

We will call the submanifolds $\overline{M}_{0,\tau}$, resp. $M_{0,\tau}$, *closed*, resp. *open* strata of the structure stratification of $\overline{M}_{0,S}$.

In particular, closed stratum $\overline{M}_{0,\sigma}$ is a substratum of another one $\overline{M}_{0,\tau}$ of relative codimension one, iff σ can be obtained from τ by inserting one extra edge in place of a vertex v of τ and distributing half edges (or tails) at v according to a two–partition.

More generally, embeddings $\overline{M}_{0,\sigma} \subset \overline{M}_{0,\tau}$ of relative codimension $d \geq 1$ are classified by subsets of edges of σ of cardinality d such that their "blowing down" produces τ. This implies that they can be obtained by iterating embeddings of codimension one.

From this discussion, it follows that

$$M_{0,\tau} = \overline{M}_{0,\tau} \setminus (\bigcup_{\sigma} \overline{M}_{0,\sigma})$$

where the union is taken over all substrata of relative codimension one, that in turn bijectively correspond to edges of τ.

2.2. Very good pairs of strata. We will call an ordered pair of locally open strata $(M_{0,\sigma}, M_{0,\tau})$ in $\overline{M}_{0,S}$ *(effective) very good pair* if $M_{0,\sigma} \subset M_{0,\tau}$, and

$$\dim M_{0,\sigma} = \dim M_{0,\tau} - 1.$$

This is the main part of the Definition 9.2.1, 2, in [HuM–St17]. In order to use very good pairs for a description of modular Nori motives, we must check that all $M_{0,\tau}$ are affine and smooth. This is well known. We will include in this picture also maximally degenerate cases with $\dim M_{0,\tau} = 0$ and $M_{0,\sigma} = \emptyset$, that is $M_{0,\tau} = M_{0,\sigma} = M_{0,S}$.

2.3. Proposition. *Every non–empty locally open stratum $M_{0,\tau}$ is the second term of a very good pair.*

Proof. This statement follows from our remark above regarding inclusions of closed strata.

2.4. Nori motives of strata. We see now that diagrams representing *only* motives of strata have a nice and compact combinatorial description.

Depending on which of the classic cohomology theory we want to focus, we will land in one of the categories $\mathcal{MM}(k)$ mentioned in Theorem 1.8.

It is essential to keep in mind the role of coefficient ring R, cf. [HuM–St17], Remark 9.1.8.

The study of what we call here "symmetries of modular Nori motives" is also based upon the Theorem 1.8 above.

This study proceeds along the same path as in Sec. 4 of [CMMar20], however to each stretch of this path we add its discussion in terms of modular Nori motives.

2.5. Incidence Hopf algebras. We start with a set of (connected) finite stable trees sufficient to encode all relevant (closed/locally closed) strata of moduli spaces $\overline{M}_{0,\tau}$ discussed above, over an algebraically closed ground field of characteristic zero. We complete it by including all finite forests f consisting of trees τ encoding such strata, and *denote by $\mathcal{O}$ the resulting set of stable graphs.*

Introduce on $\mathcal{O}$ the following partial order: $f_0 \leq f_1$, iff f_1 can be obtained from a subforest of f_0 by grafting some couples of tails of f_0 (which produces edges in f_1). Each poset obtained in this way in $\mathcal{O}$ has a unique minimal element f and a maximal element f', which is a rooted tree. Such posets are called "intervals" and denoted $[f, f']$. In particular, in the collection of posets constructed in this way from forests in $\mathcal{O}$, every interval is isomorphic to a product of maximal intervals. We denote intervals in this partial order by $[f, f']$. For more details, see [ChaLiv07].

Starting with such a collection of posets, one can construct an associated commutative Hopf algebra over $\mathbf{Q}$, the "incidence Hopf algebra", which we denote here by $\mathcal{A}_{\mathcal{O}}$. The commutative multiplication of the Hopf algebra is the product of intervals and the coproduct is given by

$$\Delta[f, f'] = \sum_{f \leq f'' \leq f'} [f, f''] \otimes [f'', f']. \tag{2.1}$$

Later it turns out that in general $\mathcal{A}_{\mathcal{O}}$ is a free commutative algebra, spanned by the isomorphism classes of products of maximal intervals. However, some products of maximal intervals with non–pairwise isomorphic factors and with different numbers of factors can be isomorphic.

For studies of geometric and Galois symmetries, it is important to connect this definition of Hopf algebra with Connes–Kreimer construction of Hopf algebra of rooted trees in [CoKr00]. This was done in Section 6.3 of [ChaLiv07].

As an algebra, this is the commutative polynomial ring generated by the rooted trees τ, in which a product of rooted trees τ_i is identified with a forest $f = \tau_1 \sqcup \cdots \sqcup \tau_n$.

In order to define the coproduct, we must first introduce the so called *admissible cuts.* One admissible cut c of a rooted tree τ is a subset $c \subset E_\tau$ (possibly empty) such that intersection of c with any path from a the root to leaf of τ contains ≤ 1 edge. Such a cut c determines a new tree $\rho_c(\tau)$ (the part of τ that remains attached to the root after the cut) and a forest $\pi_c(\tau)$ (the union of branches that are pruned by the cut) in the following way.

If $c = \emptyset$, then $\rho_c(\tau) := \tau$, and $\pi_c(\tau)$ is the empty tree. If c consists of one edge e, then in terms of geometric realisations, $\rho_c(\tau)$ is the result of cutting this edge in two halves and taking the one half containing the root of τ as $\rho_c(\tau)$, and the remaining part as $\pi_c(\tau)$. The root of $\pi_c(\tau)$ is the remaining half of e. In the case of a more general tree τ each path from the root to one of the leaves that contains a cut of c gives rise to a component of the forest $\pi_c(\tau)$.

Let $Cuts(\tau)$ be the set of admissible cuts of τ. Then the coproduct is defined by the following formula (extended multiplicatively to forests):

$$\Delta(\tau) = \sum_{C \in Cuts(\tau)} \rho_C(\tau) \otimes \pi_C(\tau), \tag{2.2}$$

One can check that it admits the antipode ω satisfying identities $\omega(1) = 1$,

$$\omega(\tau) = -m(\omega \otimes id - \iota\epsilon)\Delta(\tau)$$

where m is the multiplication, ι the unit and ϵ the counit.

2.6. Hopf algebras of modular motives. We can now construct the map from the set of Nori motives of strata to the Hopf algebra of trees, using the basic definitions in 1.2.

2.7. Groups of symmetries. Now we want to include the action of a group G upon the set of trees $\mathcal{O}$. Let $\gamma \in G$ maps τ to $\gamma\tau$.

2.7.1. ***G*–balanced cuts.** In order to incorporate the Galois action in the construction of the Hopf algebra, we can proceed as in Definition 4.3 of [CMMar20] where a notion of G–balanced cuts is considered. These are admissible cuts of τ with the property that, for each $\gamma \in G$, the pair $(\gamma\rho_C(\tau), \gamma\pi_C(\tau))$ is an admissible cut of the tree $\gamma\tau$. Denote the set of such cuts $Cuts_{G(\tau)}$.

As in [CMMar20], the action $\gamma \in G$ is the action of γ on the tree components of the forest $\pi_C(\tau)$. The Hopf algebra $\mathcal{A}_{\mathcal{O},G}$ is defined as above as a commutative algebra over $\mathbf{Q}$, with coproduct

$$\Delta(X_\tau) = \sum_{C \in Cuts_{G(\tau)}} X_{\rho_C(\tau)} \otimes X_{\pi_C(\tau)} \ . \tag{2.3}$$

2.7.2. Lemma. *If the G–action does not change combinatorics of trees themselves and acts on the labelings of the flags then all admissible cuts are G–balanced, hence $\mathcal{A}_{\mathcal{O},G} = \mathcal{A}_{\mathcal{O}}$.*

Proof. In this situation the grafting operations of trees are compatible with the G–action. Thus, for all $\gamma \in G$ we have

$$\gamma\tau_1 *_{(\gamma t_1, \gamma t_2)} \gamma\tau_2 = \gamma \cdot \tau_1 *_{(t_1,t_2)} \tau_2 \,. \tag{2.4}$$

This shows as in Lemma 4.4 of [CMMar20] that the G–balanced condition holds.

From now on, appearance of modular Nori motives adds nothing to the discussion of quantum statistical mechanics of the relevant versions of Galois and Grothendieck–Teichmüller groups in Sections 3–4 of [CMMar20], and we will stop here.

2.8. Connes–Kreimer Hopf algebras in the operadic context. After having read the first draft of this article, Bruno Vallette made the following remark that he kindly allowed us to include in the final lines of it.

The Connes–Kreimer Hopf algebra actually comes from a general construction which associates a commutative Hopf algebra to any cooperad under the assignment $\mathcal{C} \mapsto S(\bigoplus_n \mathcal{C}(n)^S_n)$, where the product is free and where the coproduct is induced by the cooperad structure. The Connes–Kreimer Hopf algebra is the one obtained from the linear dual of the operad $PreLie$. But one could consider as well the cohomology cooperad $H^*(\overline{M}_{0,n+1})$, which "loses" nothing since the topological operad $\overline{M}_{0,n+1}$ is formal. This is not what we do here, but it should give a related construction.

2.9 Nori motives and quantum statistical mechanical systems. In [LieMMar19] it was shown that the endomorphisms of the Bost–Connes quantum statistical mechanical system can be lifted to the category of Nori motives with a residually finite action of $\hat{\mathbf{Z}}$. On the other hand, in [MaMar19-2] the Bost–Connes system was extended to the non-abelian Galois theory by enriching the Bost–Connes algebra through the Drinfeld–Ihara involution. This suggests that the lifting of the Bost–Connes system to Nori motives of [LieMMar19] would extend to a lift of this enriched structure with the absolute Galois group action when considering the modular Nori motives described in the present paper.

Acknowledgment. N.C. Combe acknowledges support from the Minerva Fast track grant from the Max Planck Institute for Mathematics in the Sciences, in Leipzig. M. Marcolli acknowledges support from NSF grant DMS-1707882 and NSERC grants RGPIN-2018-04937 and RGPAS-2018-522593.

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