Moduli operad over $\mathbb{F}_1$

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Contents

1. Introduction and summary

Of many recently suggested definitions of $\mathbb{F}_1$-geometry, we work with the one developed in [19] that seems to be the minimal one. Namely, an $\mathbb{F}_1$-scheme is represented by its lift to $\mathtt{Spec}(\mathbb{Z})$ and the relevant descent data which are essentially a representation of the lifted scheme as a disjoint union of locally closed tori.

This notion of $\mathbb{F}_1$-geometry can be seen as the simplest geometrization of the condition that the class in the Grothendieck ring of the variety decomposes as a sum of classes of tori, with non-negative coefficients. This motivic condition accounts for the expected behavior of points over $\mathbb{F}_1$ and over "extensions" $\mathbb{F}_{1^m}$ in relation to the counting of points over $\mathbb{F}_{q^m}$ and zeta functions.

In this setting, we show that, while the torification condition (possibly with additional restrictions such as a compatibility with an affine covering) provides a viable notion of "algebraic variety over $\mathbb{F}_1$," when one considers possible descent data to $\mathbb{F}_1$ for stable curves of genus zero with marked points one needs to consider also objects that are analogs of "constructible sets" over $\mathbb{F}_1$, which can be seen as formal differences of torifications. In general, the complement of an algebraic variety in another need not be an algebraic variety, but it is a constructible set. Similarly, not all points or subvarieties over $\mathbb{F}_1$ (in the sense of torifications as well as in other forms of $\mathbb{F}_1$-geometry) are *complemented*. The complemented case corresponds to those $\mathbb{F}_1$-subvarieties whose complement also defines an $\mathbb{F}_1$-variety, while in the non-complemented case one obtains an $\mathbb{F}_1$-constructible set, according to a suitable notion of differences of torifications that we refer to as "constructible torifications." The moduli spaces $\overline{M}_{0,n}$ and their generalizations $T_{d,n}$ constructed in [5] also have a structure of $\mathbb{F}_1$-constructible sets. The operad structure on these moduli spaces is also compatible with the $\mathbb{F}_1$-structure and the operad morphisms give rise to $\mathbb{F}_1$-constructible morphisms.

In Section 2, we recall the notion of torification from [19] and we discuss different equivalence relations that determine when two choices of torification on the same variety over $\mathbb{Z}$ determine the same $\mathbb{F}_1$-structure. This leads to three different notions of $\mathbb{F}_1$-morphisms, which we refer to as strong, ordinary, and weak morphisms.

In Section 3 we focus on the condition that the Grothendieck class of a variety decomposes into a sum of tori with non-negative coefficients, which is necessary for the existence of geometric torifications. We show that it is satisfied for the moduli spaces $\overline{M}_{0,n}$ and $T_{d,n}$. This follows the same argument used in [24] and [5], respectively, for the computation of the Poincaré polynomials. We also show that these same computations provide a generating series for the numbers of $\mathbb{F}_{1^m}$-points of $\overline{M}_{0,n}$ and $T_{d,n}$.

In Sections 4 and 5 we discuss the notions of *complemented* points and *complemented subspaces* in $\mathbb{F}_1$-geometry. We analyze the geometric torifications of stable curves of genus zero and the role of the marked points as uncomplemented points. We introduce the notions of *constructible sets* over $\mathbb{F}_1$ and of *constructible torifications*, which are formal differences of torifications preserving the positivity of the Grothendieck class. In Section 6 we show that the moduli spaces $\overline{M}_{0,n}$ and $T_{d,n}$ are $\mathbb{F}_1$-constructible sets.

In Section 7, for each $d \geq 1$, we introduce the operads with components $\{T_{d,n+1}\}$ from [5] and we show that the operadic structure morphisms are compatible with the structure of $\mathbb{F}_1$-constructible sets. The operad composition operations and the morphisms that forget marked points determine strong $\mathbb{F}_1$-constructible morphisms, while the action of $\mathbf{S}_n$ that permutes marked points acts through ordinary $\mathbb{F}_1$-constructible morphisms. In Section 8 we also show that, if one uses the description of the moduli spaces $\overline{M}_{0,n}$ and $T_{d,n}$ as iterated blowups, related to the Fulton–MacPherson compactifications as in [5], then the projection maps of the iterated blowups are only weak $\mathbb{F}_1$-morphisms.

In Section 9 we focus on the *blueprint* approach to $\mathbb{F}_1$-geometry, developed in [21], see the chapter of Lorscheid in this volume. We make explicit a blueprint structure of $\overline{M}_{0,n}$ based upon explicit equations for $\overline{M}_{0,n}$, as in [12], [16]. We consider then the genus-zero boundary modular operad $\{\overline{M}^0_{g,n+1}\}$ whose components are, by definition, unions of those boundary strata in $\{\overline{M}_{g,n+1}\}$ that parametrize curves whose normalized irreducible components are projective lines. This is an operad in the category of DM-stacks, so that for its complete treatment within the setting of torifications it would be necessary to develop a formalism of stacky $\mathbb{F}_1$-geometry compatible with torifications as descent data. We describe a blueprint structure on the genus-zero boundary $\overline{M}^0_{g,n+1}$ of the higher-genus moduli spaces, using a crossed product construction.

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2. Torifications

The notion of torification introduced in [19] is the following condition, which we refer to in this chapter as *geometric torification*.

Definition 2.0.1 (see [19]). A *torification* of the scheme X is a morphism of schemes $e_X : T \to X$ from a disjoint union of tori $T = \coprod_{j\in I} T_i$, where $T_j = \mathbb{G}_m^{d_j}$, such that the restriction of e_X to each torus is an *immersion* (i.e. isomorphism with a locally closed subscheme), and such that e_X induces bijections of k-points, $e_X(k)\colon T(k) \xrightarrow{\sim} X(k)$, for every field k.

Moreover, in [19] the authors also consider the stronger notion of *affine torification.*

Definition 2.0.2 (see [19]). The torification e_X is called *affine* if there exists an affine covering $\{U_\alpha\}$ of X compatible with e_X in the following sense: for each affine open set U_α in the covering, there is a subfamily of tori $\{T_j \mid j \in I_\alpha\}$ in the torification e_X such that the restriction of e_X to the disjoint union of tori from this subfamily is a torification of U_α.

2.1. Levels of torified structures. We assume that X is a variety over $\mathbb{Z}$. There are three levels of increasingly restrictive conditions in this approach based on defining $\mathbb{F}_1$-structures via torifications: the basic level is a decomposition of the class in the Grothendieck ring, the second is a geometrization of this decomposition at the level of the variety itself, and the third level includes more restrictive conditions, such as affine and regular.

(1) *Torification of the Grothendieck class*: this is the weakest condition and it simply consists of the requirement that the class $[X] \in K_0(\mathscr{V}_\mathbb{Z})$ in the Grothendieck ring can be written as

$$[X] = \sum_k a_k \mathbb{T}^k, \tag{1}$$

where $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L} - 1$, and $\mathbb{L} = [\mathbb{A}^1]$ the Lefschetz motive (the class of the affine line), and with coefficients $a_k \geq 0$.

(2) *Geometric torification*: this is the condition of Definition 2.0.1 above.

(3) *Affine torification*: where the geometric torification is also affine in the sense of Definition 2.0.2.

(4) *Regular torification*: this is a geometric torification where one also requires that the closure of each torus in the torification is itself a union of tori of the torification.

Roughly, one can understand these different levels as describing stronger forms of $\mathbb{F}_1$-structures based on torification. The decomposition of the class in the Grothendieck ring reflects how one expects that $\mathbb{Z}$-varieties that descend to $\mathbb{F}_1$ should behave with respect to motivic properties such as the zeta function and counting of points. The notion of geometric torification introduced in [19] can be seen as a minimal way of making this motivic behavior "geometric." The further level, given by the affine condition, was introduced in [19], motivated by the comparison between this approach to $\mathbb{F}_1$-geometry and the approaches developed by Soulé in [27], and by Connes and Consani in [6]. However, in many respects, it would be natural to expect that varieties like Grassmannians would descend to $\mathbb{F}_1$, have natural torifications coming from their cell decompositions that are not affine. This concern justifies retaining the intermediate level of $\mathbb{F}_1$-structure given by geometric torifications *without* the affine condition. As we shall argue later, this level already provides a very rich and interesting structure. The regularity condition, which is independent of the affine requirement, but is usually considered for affine torifications, was introduced in [19] as a possible way to "rigidify" the choice of torification. We follow here a different approach based on considering different levels of equivalence relations among torifications, hence we will not consider the regularity condition.

2.2. Equivalent torifications and morphisms. When we consider *geometric torifications* as data defining $\mathbb{F}_1$-structures on $\mathbb{Z}$-varieties, one would like to have a natural equivalence relation describing when two different choices of torification on the same varieties should be regarded as defining the *same* $\mathbb{F}_1$-structure.

We first recall the notion of *torified morphism* introduced in [19].

Definition 2.2.1 (see [19]). A *morphism* of torified varieties (*torified morphism*) $\Phi\colon (X, e_X\colon T_X \to X) \to (Y, e_Y\colon T_Y \to Y)$ is a triple $\Phi = (\phi, \psi, \{\phi_i\})$, where $\phi\colon X \to Y$ is a morphism of $\mathbb{Z}$-varieties, $\psi\colon I_X \to I_Y$ is a map of the indexing sets of the two torifications, and $\phi_j : T_{X,j} \to T_{Y,\psi(j)}$ is a morphism of algebraic groups, such that $\phi \circ e_X|_{T_{X,j}} = e_Y|_{T_{Y,\psi(j)}} \circ \phi_j$.

In [19], a notion of *affinely torified morphisms* was also introduced: these are torified morphisms in the sense recalled above, between affinely torified varieties, such that, if $\{U_j\}$ is an affine open covering of X compatible with the torification, then for every j the image of U_j under Φ is an affine subscheme of Y. The following lemma, communicated to us by Lorscheid, shows that it is not necessary to assume this as an additional condition for torified morphisms between affine affinely torified varieties.

Lemma 2.2.2. *Let $\Phi\colon (X, e_X\colon T_X \to X) \to (Y, e_Y\colon T_Y \to Y)$ be a torified morphism between affinely torified varieties, with $\{U_i\}$ and $\{V_j\}$ respective affine torified coverings. Then Φ is an affinely torified morphism.*

Proof. Let $W_j = \Phi^{-1}(V_j)$ and $U_{ij} = U_i \cap W_j$. Then U_{ij} is a torified and quasi-affine subscheme of X that maps to V_j. The collection of all U_{ij} covers X. Consider then $Z_{ij} = \mathtt{Spec}\big(\mathscr{O}_X(U_{ij})\big)$. Since U_{ij} is quasi-affine, the natural map $U_{ij} \to Z_{ij}$ is an embedding of U_{ij} into the affine subscheme Z_{ij} of U_i. Moreover, the morphism $U_{ij} \to V_j$ extends naturally to a morphism $Z_{ij} \to V_j$ (since V_j is affine), which means that Z_{ij} is contained in W_j. Therefore, Z_{ij} is contained in $U_i \cap W_j = U_{ij}$, hence we have that $U_{ij} = Z_{ij}$ is affine, and so Φ is an affinely torified morphism. □

We consider the following notions of equivalence of torifications on a given $\mathbb{Z}$-variety X.

(1) *Strong equivalence*: the identity morphism is torified.

(2) *Ordinary equivalence*: there exists an isomorphism of X that is torified.

(3) *Weak equivalence*: one identifies as the same $\mathbb{F}_1$-structure two torifications on a variety X such that X has a decomposition into a disjoint union of subvarieties $X = \bigcup_j X_j$ and $X = \bigcup_j X'_j$, respectively compatible with the torifications, and such that there exist isomorphisms $\phi_i \colon X_i \to X'_i$ that are torified. One considers the equivalence relation generated by these identifications.

In the case of a weak equivalence the isomorphisms on the pieces of the decomposition do not necessarily extend to isomorphisms of the whole variety. Typical examples of this third condition are obtained by considering cell decompositions compatible with the torifications. For example, one can consider $\mathbb{P}^1 \times \mathbb{P}^1$ with the cell decomposition $\mathbb{P}^1 = \mathbb{A}^0 \cup \mathbb{A}^1$ on each factor. One can then consider the standard torification of $\mathbb{P}^1 \times \mathbb{P}^1$ compatible with the cell decomposition and the torification obtained by taking a torification of the diagonal and of its complement in the $\mathbb{A}^2$ cell, and the torification of the other cells as before. These two torifications are related by a weak equivalence, but not by an ordinary one.

The choice of the equivalence relation above determines what morphisms of $\mathbb{Z}$-varieties can be regarded as descending to $\mathbb{F}_1$.

(1) *Strong* $\mathbb{F}_1$*-morphisms* (or *strongly torified morphisms*): when geometric torifications are assumed to define the same $\mathbb{F}_1$-structure if and only if they are strongly equivalent, morphisms of $\mathbb{Z}$-varieties that define $\mathbb{F}_1$-morphisms are torified morphisms in the sense of Definition 2.2.1.

(2) *Ordinary* $\mathbb{F}_1$*-morphisms* (or *ordinarily torified morphisms*): under ordinary equivalence, $\mathbb{F}_1$-morphisms are all morphisms of $\mathbb{Z}$-varieties that become torified after composing with isomorphisms.

(3) *Weak* $\mathbb{F}_1$*-morphisms* (or *weakly torified morphisms*): under weak equivalence, $\mathbb{F}_1$-morphisms are morphisms of $\mathbb{Z}$-varieties that become torified after composition with weak equivalences.

In the following, we refer to the different cases above as a *strong*, *ordinary*, or *weak* $\mathbb{F}_1$*-structure*, or as geometric torifications in the strong, ordinary, or weak sense.

Example. Any toric variety has a natural torification by torus orbits. In [19], explicit affine torifications are constructed, and it is checked that toric morphisms are compatible with them. This shows that the Losev–Manin operad $\{\overline{L}_{0,n}\}$ in [22], [23], [25] has natural descent data to $\mathbb{F}_1$, in the strong sense of the notion of torifications and morphisms described above.

Remark 2.2.3. Considering torifications and $\mathbb{F}_1$-morphisms in the weak sense is very close to imposing only the condition of torification of Grothendieck classes, though it appears to be stronger, as our discussion of constructible torifications in §4 will illustrate.

Remark 2.2.4. Among the other existing approaches to $\mathbb{F}_1$-structures, the one based on the notion of *blueprint*, developed in [21], [20]—see also Lorscheid's chapter in the present book—does not resort to decompositions into tori, and it is a less restrictive form of $\mathbb{F}_1$-structure in the sense that every scheme of finite type admits a "blue model" of finite type over $\mathbb{F}_1$.

2.3. Categories of geometric torifications. The different notions of morphisms of torified varieties considered above lead to the following categorical formulation.

Proposition 2.3.1. *There are categories $\mathscr{GT}^s \subset \mathscr{GT}^o \subset \mathscr{GT}^w$ where the objects,* $\mathrm{Obj}(\mathscr{GT}^s) = \mathrm{Obj}(\mathscr{GT}^o) = \mathrm{Obj}(\mathscr{GT}^w)$*, are pairs $(X_{\mathbb{Z}}, \mathscr{T})$, with $X_{\mathbb{Z}}$ a variety over $\mathbb{Z}$ and $\mathscr{T} = \{T_i\}$ a geometric torification of $X_{\mathbb{Z}}$. Morphisms in $\mathscr{GT}^s$ are strong morphisms of geometrically torified spaces; morphisms in $\mathscr{GT}^o$ are ordinary morphisms of geometrically torified spaces; morphisms in $\mathscr{GT}^w$ are weak morphisms of geometrically torified spaces.*

Proof. According to our previous discussion, strong morphisms of geometrically torified spaces are the "torified morphisms" of Definition 2.2.1, hence the category $\mathscr{GT}^s$ is the category of torified varieties, as considered in [19]. Morphisms in $\mathscr{GT}^o$ are arbitrary compositions of torified morphisms and ordinary equivalences, which means that they can be written as arbitrary compositions of torified morphisms and isomorphisms of $\mathbb{Z}$-varieties. Since composition of two such morphisms will still be of the same kind, composition of morphisms is well defined in $\mathscr{GT}^o$. Morphisms in $\mathscr{GT}^s$ are also morphisms in $\mathscr{GT}^o$, but not the other way around. Similarly, morphisms in $\mathscr{GT}^w$ are arbitrary compositions of torified morphisms and weak equivalences, that is, arbitrary compositions of torified morphisms and local isomorphisms of the type described in §2.2 above. Again, composition is well defined. Morphisms in $\mathscr{GT}^s$ and morphisms in $\mathscr{GT}^o$ are also morphisms in $\mathscr{GT}^w$, but not conversely. □

3. Grothendieck classes and torifications

In this section we consider the moduli spaces $\overline{M}_{0,n}$, as well as their generalizations $T_{d,n}$ considered in [5], from the point of view of classes in the Grothendieck ring. The existence of a decomposition of the form (1) into tori, with non-negative coefficients, follows from the fact that these spaces can be realized as a sequence of iterated blowups starting from a variety that clearly admits a torification and blowing up loci that, in turn, admit torifications. The explicit form of the decomposition (1) mirrors the known formulae for the Poincaré polynomial and the Euler characteristic of [24] and [5] and can be obtained by a similar argument. The generating functions of [24] and [5] computing the Poincaré polynomials are also related to counting points over the extensions $\mathbb{F}_{1^m}$.

3.1. The class of $M_{0,n}$. A first simple observation, which will be useful in the following, is that the open stratum $M_{0,n}$ by itself cannot be torifed, since it fails the necessary condition that the class $[M_{0,n}]$ is torified by a decomposition (1) with non-negative coefficients.

Lemma 3.1.1. *The class $[M_{0,n}]$ has a decomposition into tori of the form*

$$[M_{0,n}] = \sum_{k=0}^{n-2} s(n-2,k) \sum_{j=0}^{k} \binom{k}{j} \mathbb{T}^j, \tag{2}$$

where $s(m,k)$ is the Stirling number of the first kind. In particular, the open stratum $M_{0,n}$ does not admit a geometric torification.

Proof. We can view $M_{0,n}$ as the complement of the diagonals in a product of $n-3$ copies of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, hence the class in the Grothendieck ring is given by

$$[M_{0,n}] = (\mathbb{T}-1)(\mathbb{T}-2)\cdots(\mathbb{T}-n+2) = \binom{\mathbb{T}-1}{n-3}(n-3)! = (-1)^n(1-\mathbb{T})_{n-2}, \tag{3}$$

where $(x)_m = \Gamma(x+m)/\Gamma(x)$ is the Pochhammer symbol, satisfying

$$(x)_m = \sum_{k=0}^{m} (-1)^{m-k} s(m,k) x^k,$$

with coefficients $s(m,k)$ the Stirling numbers of the first kind, namely the integers such that $(-1)^{m-k}s(m,k)$ is the number of permutations in $\mathbf{S}_m$ consisting of k cycles. Thus, we obtain

$$[M_{0,n}] = (-1)^n \sum_{k=0}^{n-2} (-1)^{n-k} s(n-2,k)(-1)^k(\mathbb{T}-1)^k,$$

which gives (2), where some of the coefficients are clearly negative. □

3.2. The class of $\overline{M}_{0,n}$ and $\mathbb{F}_{1^m}$-points. By Lemma 3.1.1, the open stratum $M_{0,n}$ by itself cannot be torified. However, when one considers the compactification $\overline{M}_{0,n}$, one finds that the condition of torification of the Grothendieck class is satisfied.

Proposition 3.2.1. *The classes $[\overline{M}_{0,n}] \in K_0(\mathscr{V}_{\mathbb{Z}})$ fit into a generating series*

$$\varphi(t) = t + \sum_{n=2}^{\infty} [\overline{M}_{0,n}] \frac{t^n}{n!} \in K_0(\mathscr{V}_{\mathbb{Z}})_{\mathbb{Q}}[\![t]\!] \tag{4}$$

where we write $K_0(\mathscr{V}_{\mathbb{Z}})_{\mathbb{Q}} = K_0(\mathscr{V}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and where $\varphi(t)$ is the unique solution in $t + t^2 K_0(\mathscr{V}_{\mathbb{Z}})_{\mathbb{Q}}[\![t]\!]$ of the differential equation

$$\big(1 + \mathbb{L}t - \mathbb{L}\varphi(t)\big)\varphi'(t) = 1 + \varphi(t). \tag{5}$$

In particular, the classes $[\overline{M}_{0,n}]$ *satisfy the recursive relation*

$$[\overline{M}_{0,n+2}] = [\overline{M}_{0,n+1}] + \mathbb{L} \sum_{i+j=n+1, i\geq 2} \binom{n}{i} [\overline{M}_{0,i+1}][\overline{M}_{0,j+1}], \tag{6}$$

and therefore have a decomposition (1) *with non-negative coefficients.*

Proof. The argument is analogous to the proof of [24, Theorem 0.3.1] computing the Poincaré polynomials of $\overline{M}_{0,n}$. In fact, the same argument used in [24] to determine the Poincaré polynomials applies to the computation of the Grothendieck classes, using the classes of all the $M_{0,k}$ given in (3), which we rewrite as

$$[M_{0,k}] = \binom{\mathbb{L}-2}{k-3}(k-3)!$$

which is the direct analog of [24, equation (1.2)] for the Poincaré polynomials. The existence of a decomposition (1) with non-negative coefficients then follows inductively from the fact that the classes satisfy the recursive relation (6), which follows from (5) as in [24, Corollary 0.3.2], and from the fact that the first terms of the recursion can be seen explicitly to have non-negative coefficients. □

It would be interesting to know if the Chern class of $\overline{M}_{0,n}$ also satisfies a similar recursive formula and positivity property.

Remark 3.2.2. The Poincaré polynomial for $\overline{M}_{0,n}$ can be recovered from the Grothendieck class by formally replacing $\mathbb{L}$ with q^2 in the resulting expression. This fact holds more generally for smooth projective varieties whose class in the Grothendieck ring is a polynomial $[X] = \sum_k b_k \mathbb{L}^k$ in the class $\mathbb{L}$ of the Lefschetz motive. In fact, in this case the Hodge–Deligne polynomial $h_X(u,v) = \sum_{p,q}(-1)^{p+q} h^{p,q}(X_{\mathbb{C}}) u^p v^q$ is given by $h_X(u,v) = \sum_k b_k (uv)^k$, which implies that $X_{\mathbb{C}}$ is Hodge–Tate, namely $h^{p,q}(X_{\mathbb{C}}) = 0$ for $p \neq q$. This in turn implies that the Poincaré polynomial is given by $P_X(q) = \sum_k b_k q^{2k}$, hence it is obtained from the expression for $[X]$ by formally replacing $\mathbb{L}$ by q^2.

The expression of Proposition 3.2.1 for the Grothendieck classes $[\overline{M}_{0,n}]$ can also be interpreted as giving the counting of points over "extensions" $\mathbb{F}_{1^m}$. In fact, the number of points over $\mathbb{F}_1$ can be obtained as the limit as $q \to 1$ of the function $N_X(q)$ that counts points over finite fields $\mathbb{F}_q$, possibly normalized by a power of $q-1$. The value $N_X(1)$ for a polynomially-countable variety coincides with its Euler characteristic. Similarly, one can make sense of the number of points over $\mathbb{F}_{1^m}$ as the values $N_X(m+1)$, see [7, Theorem 4.10] and [9, Theorem 1].

Proposition 3.2.3. *Let* $p_{n,m}$ *denote the number of points of* $[\overline{M}_{0,n}]$ *over* $\mathbb{F}_{1^m}$. *The generating function*

$$\varphi_m(t) = \sum_{n\geq 1} p_{n,m} \frac{t^n}{n!}$$

is a solution of the differential equation

$$\big(1 + (m+1)t - (m+1)\varphi_m(t)\big)\varphi_m'(t) = 1 + \varphi_m(t).$$

Proof. Let X be a smooth projective variety over $\mathbb{Z}$ whose class in the Grothendieck group can be written as $[X] = \sum_i a_i \mathbb{L}^i$ with the a_i non–negative integers. For all but finitely many primes p and $q = p^r$, the function that counts points of X over a finite field $\mathbb{F}_q$ is then given by $N_X(q) = \sum_i a_i q^i$. Thus, we obtain the values $N_X(m+1)$ counting $\mathbb{F}_{1^m}$-points by formally replacing $\mathbb{L}$ with $m+1$ in the expression for the Grothendieck class. □

3.3. The moduli spaces $T_{d,n}$. We consider here a family of varieties $T_{d,n}$ constructed in [5], which are natural generalizations of the moduli spaces $\overline{M}_{0,n}$.

We recall the construction of [5] of $T_{d,n}$ as a family of varieties whose points parametrize stable n-pointed rooted trees of projective spaces $\mathbb{P}^d$. They generalize the moduli spaces $\overline{M}_{0,n}$, with the latter given by $T_{1,n} = \overline{M}_{0,n+1}$. These varieties are also closely related to the Fulton–MacPherson compactifications $X[n]$ of configuration spaces [10], in the sense that for any choice of a smooth complete variety X of dimension d, one can realize $T_{d,n}$ in a natural way as a subscheme of $X[n]$.

3.4. n-pointed rooted trees of projective spaces. A *graph* τ consists of the data $(F_\tau, V_\tau, \partial_\tau, j_\tau)$ as follows: a set of flags (half-edges) F_τ; a set of vertices V_τ; boundary maps $\partial_\tau \colon F_\tau \to V_\tau$ that associate to each flag its boundary vertex; and finally the involution $j_\tau \colon F_\tau \to F_\tau$, $j_\tau^2 = 1$ that registers the matching of half-edges forming the edges of τ. We consider here only graphs whose geometric realizations are trees, i.e. they are connected and simply connected.

A structure of rooted tree is defined by the choice of a root tail $f_\tau \in F_\tau$, $j_\tau(f_\tau) = f_\tau$. Its vertex $v_\tau := \partial_\tau(f_\tau)$ also may be called the *root*.

We define the canonical orientation on the rooted trees: the root tail is oriented away from its vertex (so it is the *output*); all other flags are oriented towards the root vertex. The remaining tails are called *inputs*.

The output tail of a tree can be grafted to an input tail of another tree.

We say that a vertex v is a *mother* for a vertex v' if v' lies on an oriented path from v to the root vertex v_0 and the oriented path from v to v' consists of a single edge.

Given an oriented rooted tree τ, we assign to each vertex $v \in V_\tau$ a variety $X_v \simeq \mathbb{P}^d$. To the unique outgoing tail at v we assign a choice of a hyperplane $H_v \subset X_v$. To each incoming tail f at v we assign a point $p_{v,f}$ in X_v such that $p_{v,f} \neq p_{v,f'}$ for $f \neq f'$ and we require $p_{v,f} \notin H_v$, for all f at v.

We think of an oriented rooted tree τ, with S_τ the finite set of incoming tails of τ of cardinality n, as an n-ary operation that starts with the varieties $X_{v_i} \simeq \mathbb{P}^d$ attached to the input vertices v_i, $i = 1, \ldots, m \leq n$, and glues the hyperplane $H_{v_i} \subset X_{v_i}$ to the exceptional divisor of the blowup of X_{w_i} at the point p_{w_i,f_i} where w_i is the target vertex of the unique outgoing edge of v_i and f_i is the flag of this edge with $\partial_\tau(f_i) = w_i$, ingoing at w_i. The operation continues in this way at the next step, by gluing the hyperplanes H_{w_i} to the exceptional divisor of the blowups of the projective spaces of the following vertex. At each vertex that has an incoming tail, the corresponding variety acquires a marked point. The variety obtained by this series of operations, when one reaches the root vertex, is the output of τ. It is endowed with n marked points from the incoming tails and with a given hyperplane from the outgoing tail at the root. In the terminology of [5], the output X_τ of an oriented rooted tree τ with n incoming tails is an *n-pointed rooted tree of d-dimensional projective spaces*.

The *stability condition* for X_τ is the requirement that each component of X_τ contains at least two distinct markings, which can be either marked points or exceptional divisors. By [5, Proposition 2.0.5], this condition is equivalent to the absence of nontrivial automorphisms of $\mathbb{P}^d$ fixing a hyperplane pointwise, that is, translations and homotheties in $\mathbb{A}^d$.

The variety $T_{d,n}$ is defined in [5, Theorem 3.4.4] as the moduli space of n-pointed stable rooted trees of d-dimensional projective spaces.

3.5. The class of $T_{d,n}$ and $\mathbb{F}_{1^m}$-points. In [5], the Poincaré polynomials of the varieties $T_{d,n}$ are computed, generalizing the result of [24] on the Poincaré polynomial of the moduli spaces $\overline{M}_{0,n}$. Again, the classes $[T_{d,n}]$ in the Grothendieck ring can be computed with the same technique, which shows that they satisfy the torification condition. One also obtains the counting of points over $\mathbb{F}_{1^m}$.

Proposition 3.5.1. (1) *For fixed d, the classes $[T_{d,n}] \in K_0(\mathscr{V}_\mathbb{Z})$ form a generating function*

$$\psi(t) = \sum_{n\geq 1} [T_{d,n}] \frac{t^n}{n!} \tag{7}$$

in $K_0(\mathscr{V}_\mathbb{Z})_\mathbb{Q}[\![t]\!]$, which is the unique solution in $t + t^2 K_0(\mathscr{V}_\mathbb{Z})_\mathbb{Q}[\![t]\!]$ of the differential equation

$$\big(1 + \mathbb{L}^d t - \mathbb{L}[\mathbb{P}^{d-1}]\psi(t)\big)\psi'(t) = 1 + \psi(t), \tag{8}$$

where $[\mathbb{P}^{d-1}] = \frac{\mathbb{L}^d - 1}{\mathbb{L}-1}$.

(2) *The classes $[T_{d,n}] \in K_0(\mathscr{V}_\mathbb{Z})$ have a decomposition* (1) *with non-negative coefficients.*

(3) *For a fixed d, denote by $p_{n,m}$ the number of points of $T_{d,n}$ over $\mathbb{F}_{1^m}$ and form the generating function*

$$\eta_m(t) = \sum_{n\geq 1} \frac{p_{n,m}}{n!} t^n.$$

This function is a solution of the differential equation

$$(1 + (m+1)^d t - (m+1)\kappa_d(m+1)\eta_m)\eta_m' = 1 + \eta_m,$$

with $\kappa_d(q^2) = \frac{q^{2d}-1}{q^2-1}$.

Proof. (1) Thanks to [5, Theorem 5.0.2 and Corollary 5.0.3] we know that, for a fixed d and for $n \geq 2$, the generating series

$$\psi(q,t) = \sum_{n\geq 1} \frac{P_n(q)}{n!} t^n,$$

for the Poincaré polynomials $P_n(q) := P_{T_{d,n}}(q)$, with $P_1(q) = 1$, is the unique solution in $t + t^2\mathbb{Q}[q][\![t]\!]$ to the differential equation

$$(1 + q^{2d}t - q^2\kappa_d(q^2)\psi)\partial_t\psi = 1 + \psi,$$

where $\kappa_d(q^2)$ is the Poincaré polynomial of $\mathbb{P}^{d-1}$. This result is obtained using the description of the varieties $T_{d,n}$ as iterated blowups, given in [5, Theorem 3.6.2]. The same construction of $T_{d,n}$, using the blowup formula for the Grothendieck class,

$$[\mathrm{Bl}_Y(X)] = [X] + [Y]\big([\mathbb{P}^{\mathrm{codim}_X(Y)-1}] - 1\big), \tag{9}$$

gives an analogous result for the classes. Namely, the relation

$$P_{n+1}(q) = (\kappa_{d+1} + nq^2\kappa_{d-1})P_n(q) + q^2\kappa_d \sum_{i+j=n+1, 2\le i\le n-1} \binom{n}{i} P_i(q)P_j(q)$$

satisfied by the Poincaré polynomials, as shown in [5], is replaced by the analogous relation for the Grothendieck classes

$$[T_{d,n+1}] = \big([\mathbb{P}^d] + n\mathbb{L}[\mathbb{P}^{d-2}]\big)[T_{d,n}] + \mathbb{L}[\mathbb{P}^d] \sum_{i+j=n+1, 2\le i\le n-1} \binom{n}{i} [T_{d,i}][T_{d,j}]. \tag{10}$$

These relations, for Poincaré polynomials and Grothendieck classes, respectively, can be seen from the inductive presentation of the Chow group and the motive of $T_{d,n}$ given in [5, §4], with (10) following from the formula for the motive given in [5, Theorem 4.1.1].

(2) The existence of a decomposition (1) of $[T_{d,n}]$ with non-negative coefficients then follows from the fact that these classes satisfy the recursive relation (10), analogous to (6), which can be used to prove the statement inductively, as in the case of $\overline{M}_{0,n}$.

(3) The counting $N_{T_{d,n}}(m+1)$ of $\mathbb{F}_{1^m}$-points is obtained, as in the case of $\overline{M}_{0,n}$, by formally replacing $\mathbb{L}$ with $m+1$ in the expression for the Grothendieck classes, or equivalently by replacing q^2 with $m+1$ in the Poincaré polynomial. □

4. Complemented subspaces and constructible sets

In Borger's approach to $\mathbb{F}_1$-geometry via Λ-rings, [2], one has a notion of *complemented* $\mathbb{F}_1$*-points*. Namely, a sub-Λ-space $Y \subset X$ is *complemented* if the complement $X \setminus Y$ admits a Λ-space structure so that the map $X \setminus Y \hookrightarrow X$ is a morphism of Λ-spaces. In the case of toric varieties, with the Λ-space structure determined by the torus orbits, the complemented subspaces are unions of closures of torus orbits. In particular, the "complemented $\mathbb{F}_1$-points" are the fixed points of the torus action, whose number equals the Euler characteristic.

The approach to $\mathbb{F}_1$-geometry via torifications is weaker than the approach via Λ-rings. For example, as observed in [2, Example 2.8], with the exception of projective spaces, flag varieties do not admit a Λ-space structure, though they certainly admit (non-affine) geometric torifications. However, it is possible to consider an analogous notion of complemented subspaces in the setting of torifications.

4.1. Complemented $\mathbb{F}_1$-points and torifications. Our use, in the previous section, of the decomposition into tori of the class in the Grothendieck ring in order to count $\mathbb{F}_{1^m}$-points is based on thinking, as in [7], [9], of this counting as being given by the values $N_X(m+1)$ of the polynomial $N_X(q)$ counting points over $\mathbb{F}_q$. In terms of Grothendieck classes, we obtained the counting of $\mathbb{F}_{1^m}$-points as

$$\#X(\mathbb{F}_{1^m}) = [X]|_{\mathbb{T}=m} = \sum_k a_k m^k, \tag{11}$$

by formally replacing the variable $\mathbb{T}$ with m in the expression $[X] = \sum_k a_k \mathbb{T}^k$, with $a_k \geq 0$ for the Grothendieck class. The case of points over $\mathbb{F}_1$ corresponds to $m = 0$, with $\#X(\mathbb{F}_1) = [X]|_{\mathbb{T}=0} = a_0 = \chi(X)$. Essentially, this means that, for a variety X with a torification, only the 0-dimensional points contribute to $\mathbb{F}_1$-points, while each k-dimensional torus T^k of the torification with $k > 0$ contributes m points over $\mathbb{F}_{1^m}$ for each $m \geq 1$. This is related to the general philosophy that the extensions $\mathbb{F}_{1^m}$ are related to actions of the groups μ_m of mth roots of unity, see [15] and more recently [8], [25].

This leads to a natural generalization of the notion of complemented $\mathbb{F}_1$-points in the context of torifications. The counting formula (11) implies that, according to this notion, $\mathbb{F}_1$-points are points of X such that the Grothendieck class of the complement of these points still admits a decomposition into tori with non-negative coefficients. At the level of geometric torifications, it is natural therefore to introduce a stronger notion of complemented points as follows.

Definition 4.1.1. Let X be a variety over $\mathbb{Z}$, with a geometric torification. A finite set of points S is strongly (resp. ordinarily, weakly) *complemented* if the complement $X \setminus S$ also has a geometric torification, such that the inclusion $X \setminus S \hookrightarrow X$ is a strongly (resp. ordinarily, weakly) torified morphism.

For example, if we consider $\mathbb{P}^1$ with a torification given by the choice of two points, each of these two points is a strongly complemented $\mathbb{F}_1$-point, while any other point would be an ordinarily complemented $\mathbb{F}_1$-point, since the complement can be torified and the inclusion becomes a torified morphism after composing with an isomorphism of $\mathbb{P}^1$.

4.2. Complemented torifications. Similarly, one has a notion of *complemented subspace* in a torified variety. Torifications behave well with respect to blowups along complemented subspaces.

Definition 4.2.1. Let X be a variety over $\mathbb{Z}$ with a geometric torification. A subvariety $Y \subset X$ is said to be strongly (resp. ordinarily, weakly) *complemented* if both Y and the complement $X \setminus Y$ have a geometric torification, so that the inclusions $Y \hookrightarrow X$ and $X \setminus Y \hookrightarrow X$ are strongly (resp. ordinarily, weakly) torified morphisms.

On a variety that has a geometric torification compatible with a Λ-structure, the complemented condition for sub-Λ-spaces of [2] implies the strong form of complementation of Definition 4.2.1.

Example. Consider $\mathbb{P}^n$ with a torification $\mathscr{T} = \{T_i\}$ and $\mathbb{P}^n \times \mathbb{P}^n$ with the torification $\{T_i \times T_j\}$. The diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ is weakly complemented, but neither ordinarily nor strongly complemented. In the big cell $\mathbb{A}^n \times \mathbb{A}^n$ with the product torification, the diagonal is ordinarily complemented but not strongly complemented.

We have the following behavior of torifications with respect to blowups.

Proposition 4.2.2. *Let X be a variety over $\mathbb{Z}$ with a geometric torification and let $Y \subset X$ be a strongly (resp. ordinarily, weakly) complemented subspace. Then the torifications of Y and of $X \setminus Y$ for which the inclusions are strongly (resp. ordinarily, weakly) torified morphisms determine a geometric torification of the blowup $\mathrm{Bl}_Y(X)$ of X along Y, such that the morphism $\pi\colon \mathrm{Bl}_Y(X) \to X$ is strongly (resp. ordinarily, weakly) torified.*

Proof. It suffices to show that the strongly (resp. ordinarily, weakly) compatible torifications of X, Y determine a torification of the exceptional divisor of the blowup, since the complement is then torified by the torification of $X \setminus Y$. Thus, we consider the projectivized normal bundle $\mathbb{P}\big(\mathscr{N}_X(Y)\big)$. The restriction of the bundle over the tori of the torification of Y is trivial, hence $\mathbb{P}\big(\mathscr{N}_X(Y)\big)$ can be torified by the products of the tori in the torification of Y with the tori in a torification of $\mathbb{P}^{\mathrm{codim_X(Y)}-1}$. □

The blowup operation does not behave well with respect to geometric torifications in the non-complemented case. For example, the blowup of a 2-dimensional torus at a point does not have a torification compatible with the blowup morphism, even in the weak sense.

4.3. Geometric torifications of stable curves of genus zero. The fibers of the forgetful morphism $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ consist of the stable curves of genus zero. We show that these admit geometric torifications. In general, these torifications are neither regular nor affine.

Lemma 4.3.1. *Let C be a stable (pointed) curve of genus zero. A choice of a rooted tree, for a torification of $\mathbb{P}^1$, and of a point in each other component determines a geometric torification of C.*

Proof. A geometric torification of C is obtained by identifying C with a tree of $\mathbb{P}^1$ with marked points, choosing a root vertex and a torification of the $\mathbb{P}^1$ at the root, for which the two 0-dimensional tori are away from the intersection points with other components (and from the additional marked points), and then by choosing at each adjacent vertex a torification given by a torification of the complement of the intersection point with the $\mathbb{P}^1$ of the root vertex, with the 0-dimensional torus chosen away from the intersection points with other $\mathbb{P}^1$'s (and away from the additional marked points), and so on. □

These torifications reflect the decomposition into tori of the class $[C]$ in the Grothendieck ring: a tree of $\mathbb{P}^1$ with N vertices has class $[C] = N\mathbb{T} + N + 1$.

4.4. Marked points, $\mathbb{F}_{1^m}$-points, and constructible sets. When considering stable curves of genus zero with n marked points, the geometric torifications considered in Lemma 4.3.1 do not reflect the presence of the marked points, as these are not part of the torification. However, in order to descend to $\mathbb{F}_1$ the notion of marked points, we need to ensure that the stable curve C has enough points, possibly after passing to an extension to some $\mathbb{F}_{1^m}$.

Lemma 4.4.1. *If C is a tree of $\mathbb{P}^1$ with N vertices, then the number of points of C over $\mathbb{F}_{1^m}$ is $N(m+1)+1$.*

Proof. This follows from the same argument used before, computing the number of points over $\mathbb{F}_{1^m}$ from the Grothendieck class by formally replacing $\mathbb{L} = \mathbb{T}+1$ with $m+1$. □

In particular, in the case of a single $\mathbb{P}^1$ the number of points over $\mathbb{F}_{1^m}$ is $m+1$. Thus, when we consider stable curves of genus zero with n marked points, one should work with an extension $\mathbb{F}_{1^m}$ with $m \geq n-1$. Passing to an extension in this way is necessary in order to have a morphism induced by the action of $\mathbf{S}_n$ that permutes points. We will discuss more precisely the nature of such morphisms below. This phenomenon is similar to what happens in [6], where Chevalley groups define varieties over the extension $\mathbb{F}_{1^2}$. Notice that, even after considering $\mathbb{F}_{1^m}$-points so as to ensure the existence of the correct number of marked points, one needs to work with points that are not necessarily complemented.

In the usual case of algebraic varieties, the complement of an algebraic variety inside another may not determine an algebraic variety, but a *constructible set.* When we consider stable curves of genus zero with marked points, the fact that the points are not complemented means that the complement does not define an $\mathbb{F}_1$-variety. However, one expects that it will define an $\mathbb{F}_1$-constructible set, in an appropriate sense. We show in the rest of this section how one can define a suitable notion of $\mathbb{F}_1$-constructible sets, by relaxing the notion of geometric torification, while retaining intact the decomposition of the class in the Grothendieck ring.

5. Constructible sets over $\mathbb{F}_1$ and torifications

The need to consider points that are not complemented in the case of the torifications of stable curves of genus zero, suggests that one should take into consideration a further level of structure that lives in between the coarse condition about the decomposition of the Grothendieck class into a sum of tori with non-negative coefficients and the geometric torifications, which allows for a larger class of complemented subspaces and provides a suitable notion of $\mathbb{F}_1$-constructible set.

Starting from the observation that torifications behave well with respect to products and disjoint unions, but not with respect to complements, one can introduce a weaker notion of *constructible torification*, which is similar conceptually to the usual way of passing from a semigroup to a group. Recall that every constructible subset C of an algebraic variety determines a class in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{Z}})$ of varieties.

Definition 5.0.2. Let $\mathscr{C}_{\mathbb{F}_1}$ be the class of constructible sets over $\mathbb{Z}$ that can be obtained, starting from $\mathbb{G}_m$, through the operations of products, disjoint unions, and complements. Let X be a constructible set over $\mathbb{Z}$. A *constructible torification* of X is a morphism of constructible sets $e_X : C \to X$ from an element $C \in \mathscr{C}_{\mathbb{F}_1}$ to X such that the restriction of e_X to each component of C is an immersion and e_X induces a bijection of k-points, $e_X(k) : T(k) \xrightarrow{\sim} X(k)$, for every field k. An $\mathbb{F}_1$*-constructible set* is a constructible set over $\mathbb{Z}$ together with a constructible torification, with the property that the class $[X]$ in the Grothendieck ring of varieties $K_0(\mathscr{V}_{\mathbb{Z}})$ has a decomposition (1) in classes of tori, with non-negative coefficients.

The class $\mathscr{C}_{\mathbb{F}_1}$ considered above includes all tori, as well as all the complements of disjoint unions of tori inside other tori, all products of such sets, and so on. $\mathbb{F}_1$-constructible sets are built out of these building blocks, with the requirement that the positivity condition on the tori decomposition of the Grothendieck class holds.

As in the case of geometric torifications, one assigns an equivalence relation between constructible torifications that corresponds to defining the same structure of $\mathbb{F}_1$-constructible sets. This can be done in a strong, ordinary and weak form, following the analogous definitions for geometric torifications.

Definition 5.0.3. Let X and Y be constructible sets over $\mathbb{Z}$, endowed with constructive torifications. A morphism $f : X \to Y$ is said to be a *strong morphism* of $\mathbb{F}_1$-constructible sets if for each component C_j of the constructible torification of Y, $f^{-1}(C_j)$ is a disjoint union of components of the partial torification of X. Let X be a variety over $\mathbb{Z}$. Two constructible torifications of X are *strongly equivalent* if the identity on X is a strong morphism of $\mathbb{F}_1$-constructible sets. They are *ordinarily equivalent* if there is an isomorphism of X that is a strong morphism of $\mathbb{F}_1$-constructible sets, and they are *weakly equivalent* if there are decompositions $\{Z_k\}$ and $\{Z'_k\}$ of X compatible with the constructible torifications, and isomorphisms $\phi_k : Z_k \to Z'_k$ that are strong morphisms of $\mathbb{F}_1$-constructible sets. An *ordinary morphism* of $\mathbb{F}_1$-constructible sets is a morphism $f : X \to Y$ such that $\psi \circ f \circ \phi$ is a strong morphism of $\mathbb{F}_1$-constructible sets, for some isomorphism ϕ of X and some isomorphism ψ of Y. A *weak morphism* of $\mathbb{F}_1$-constructible sets is a morphism $f : X \to Y$ such that $\psi_k \circ f \circ \phi_\ell$ is a strong morphism, where ψ_k and ϕ_ℓ are isomorphisms of pieces of decompositions of Y and X, respectively, compatible with the constructible torification.

The following result on blowups for constructive torifications will be useful later.

Lemma 5.0.4. *Let X be a variety over $\mathbb{Z}$ with a constructible torification. Let $Y \subset X$ be a closed subvariety, such that $X \setminus Y$ has a constructible torification and Y has a geometric torification and the inclusions are strong* (*resp. ordinary, weak*) *morphisms of constructibly torified spaces. Then these torifications determine a constructible torification of the blowup* $\mathrm{Bl}_Y(X)$ *so that the map* $\pi : \mathrm{Bl}_Y(X) \to X$ *is a strong* (*resp. ordinary, weak*) *morphism of constructibly torified spaces.*

Proof. The argument is as in Proposition 4.2.2. The exceptional divisor, which we identify with $\mathbb{P}(\mathscr{N}_X(Y))$, has a geometric torification, since it is trivial when restricted to the tori of the geometric torification of Y. The constructible torification of $X \setminus Y$ extends the torification of the exceptional divisor to a constructible torification of $\mathrm{Bl}_Y(X)$. □

For this construction to extend to the case where the blowup locus Y has a constructible torification, one would need to ensure that the bundle $\mathbb{P}(\mathscr{N}_X(Y))$ is trivial when restricted to the components $C_i \in \mathscr{C}_{\mathbb{F}_1}$ of the decomposition of Y. This is the case, for instance, when the complements of unions of tori inside other tori in the sets C_i extend to actual (not necessarily disjoint) tori in Y.

5.1. Categories of constructible torifications. As in the case of geometric torifications, the different notions of morphisms of constructible torifications give rise to different categories.

Proposition 5.1.1. *There are categories $\mathscr{C}\mathscr{T}^s \subset \mathscr{C}\mathscr{T}^o \subset \mathscr{C}\mathscr{T}^w$ where the objects* $\mathrm{Obj}(\mathscr{C}\mathscr{T}^s) = \mathrm{Obj}(\mathscr{C}\mathscr{T}^o) = \mathrm{Obj}(\mathscr{C}\mathscr{T}^w)$ *are pairs $(X_{\mathbb{Z}}, \mathscr{C})$, where $X_{\mathbb{Z}}$ is a constructible set over $\mathbb{Z}$ and $\mathscr{C} = \{C_i\}$ is a constructible torification of $X_{\mathbb{Z}}$, in the sense of Definition* 5.0.2. *Morphisms in $\mathscr{C}\mathscr{T}^s$ are strong morphisms of constructibly torified spaces; morphisms in $\mathscr{C}\mathscr{T}^o$ are ordinary morphisms of constructibly torified spaces; morphisms in $\mathscr{C}\mathscr{T}^w$ are weak morphisms of constructibly torified spaces.*

Proof. Strong morphisms of constructibly torified spaces are as in Definition 5.0.3. The torified condition is preserved by composition. Morphisms in $\mathscr{C}\mathscr{T}^o$ are arbitrary compositions of strong morphisms and isomorphisms of $\mathbb{Z}$-constructible sets, hence composition is also well-defined. These are the ordinary morphisms of constructibly torified spaces, as in Definition 5.0.3. Morphisms in $\mathscr{C}\mathscr{T}^s$ are also morphisms in $\mathscr{C}\mathscr{T}^o$, but in general not conversely. Similarly, morphisms in $\mathscr{C}\mathscr{T}^s$ are arbitrary compositions of strong morphisms and weak equivalences, in the sense of Definition 5.0.3. Composition is well defined and the morphisms in $\mathscr{C}\mathscr{T}^s$ and in $\mathscr{C}\mathscr{T}^o$ between any pair of objects form a proper subset of the morphisms in $\mathscr{C}\mathscr{T}^w$ between the same objects. □

6. Constructible torifications of moduli spaces

We apply the notion of $\mathbb{F}_1$-constructible sets introduced above in order to define $\mathbb{F}_1$-structures on the moduli spaces $\overline{M}_{0,n}$ and on their generalizations $T_{d,n}$.

6.1. Constructible torification of $\overline{M}_{0,n}$. As we have seen, stable curves of genus zero with marked points are $\mathbb{F}_1$-constructible sets. It is therefore natural to seek a realization of the moduli spaces $\overline{M}_{0,n}$ in $\mathbb{F}_1$-geometry as $\mathbb{F}_1$-constructible sets. We show that the moduli spaces $\overline{M}_{0,n}$ have a constructible torification, underlying the decomposition of the Grothendieck class into tori.

Theorem 6.1.1. *The moduli spaces $\overline{M}_{0,n}$ are $\mathbb{F}_1$-constructible sets with a constructible torification determined by the choice of a constructible torification of $\mathbb{P}^1$ minus three points.*

Proof. For $n \geq 4$, we identify $M_{0,n}$ with the complement of the diagonals in the product of $n-3$ copies of $\mathbb{P}^1$ minus three points. The complement of three points in $\mathbb{P}^1$ is an $\mathbb{F}_1$-constructible set, with a constructible torification given by two points and the complement of one point in a 1-dimensional torus. The product of $n-3$ copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in turn has the product constructible torification. When we remove the diagonals, this corresponds to taking complements of sets in the class $\mathscr{C}_{\mathbb{F}_1}$ inside other sets in the same class, hence we still obtain a set in $\mathscr{C}_{\mathbb{F}_1}$. This does *not* define a structure of $\mathbb{F}_1$-constructible sets on the open stratum $M_{0,n}$ by itself, because the positivity condition on the class $[M_{0,n}]$ is not satisfied. However, we consider the constructible torification of $M_{0,n}$ together with the constructible torifications obtained in this same way on all products $\prod_i M_{0,k_i+1}$ with $\sum_i k_i = n$, of the lower-dimensional strata and we obtain a constructible torification of $\overline{M}_{0,n}$, which also satisfies the positivity condition on the Grothendieck class, by Proposition 3.2.1. Thus, $\overline{M}_{0,n}$ is an $\mathbb{F}_1$-constructible set. □

6.2. Constructible torification of $T_{d,n}$. We extend here the construction of geometric torifications described above for $\overline{M}_{0,n}$ to the case of the $T_{d,n}$ of [5].

We have seen that n-pointed stable curves of genus zero define $\mathbb{F}_1$-constructible sets by a choice of a geometric torification of the underlying tree of $\mathbb{P}^1$, the constructible torification given by taking the complement of the marked points in the geometric torification. One has an analogous construction for the n-marked stable trees of projective spaces described in §3.4.

Proposition 6.2.1. *Let Γ be an n-marked stable tree of projective spaces. Then Γ defines an $\mathbb{F}_1$-constructible set with a constructible torification determined by the choice of a torification $\mathscr{T}_{\mathbb{P}^d}$.*

Proof. Given an oriented rooted tree τ, at the root vertex v_0 we assign a choice of a torification $\mathscr{T}_{\mathbb{P}^d} = \{T_i\}$ of $X_{v_0} \simeq \mathbb{P}^d$, with a compatible torification of the hyperplane H_{v_0} at the unique outgoing tail at v_0. We then replace the torification $\mathscr{T}_{\mathbb{P}^d}$ of X_{v_0} by a constructible torification of the complement in X_{v_0} of the marked points corresponding to all the incoming tails at v_0, by replacing the tori T_i of the torification that contain a subset $\{p_{v_0,f,i}\}$ of the marked points with the sets $C_i \in \mathscr{C}_{\mathbb{F}_1}$ given by the complements $C_i = T_i \setminus \bigcup p_{v_0,f,i}$.

By construction, the marked points on X_{v_0} are distinct and not contained in the hyperplane H_{v_0}. We then consider the blowup of X_{v_0} at the marked points corresponding to incoming edges of the tree. By Lemma 5.0.4, the constructible torification of the complement of the marked points determines a constructible torification on the blowup. One then considers the adjacent vertices v_i, and glues the hyperplanes H_{v_i} to the exceptional divisor of the blowup of X_{v_0} at the tail mark of the edge from v_i to v_0, so as to match the torification of H_{v_i} with a piece of the torification of the exceptional divisor. One continues in this way for all the other vertices. □

We also obtain a constructible torification of $T_{d,n}$, by a construction similar to what we have for $\overline{M}_{0,n}$.

Theorem 6.2.2. *The moduli spaces $T_{d,n}$ are $\mathbb{F}_1$-constructible sets with a constructible torification determined by the choice of a constructible torification of $\mathbb{A}^d$ minus two points.*

Proof. The open stratum $TH_{d,n}$ of $T_{d,n}$ is the configuration space of n distinct points in $\mathbb{A}^d$ up to translation and homothety, or equivalently of all embeddings of a hyperplane H and n distinct points not on the hyperplane in $\mathbb{P}^d$, up to projective automorphisms that pointwise fix H. Fixing two points in $\mathbb{A}^d$ suffices to fix the symmetries, since fixing the origin $\mathbf{0} = (0, \dots, 0)$ eliminates translations and fixing another point, for instance $\mathbf{1} = (1, \dots, 1)$, takes care of homotheties. Thus, we can identify the open stratum with

$$TH_{d,n} \simeq (\mathbb{A}^d \setminus \{\mathbf{0}, \mathbf{1}\})^{n-2} \setminus \Delta, \tag{12}$$

the complement of all the diagonals Δ in the product of $n-2$ copies of $\mathbb{A}^d$ minus two points.

In the case where $d = 1$, this gives back the usual description of $M_{0,n+1} = TH_{d,n}$ as the complement of the diagonals in a product of copies of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. A choice of a constructible torification on the complement of two points in $\mathbb{A}^d$ then determines constructible torifications on the products, on the diagonals, and on the complements. As shown in [5], the compactification $T_{d,n}$ has boundary components isomorphic to products $T_{d,n_1} \times T_{d,n_2}$ with $n_1 + n_2 = n$. Thus, by considering constructible torifications on all the open strata

$$\prod_i TH_{d,n_i} \text{ with } \sum_i n_i = n,$$

we obtain a constructible torification of $T_{d,n}$. The condition of positivity of the Grothendieck class is not in general satisfied by the individual $TH_{d,k}$ and their products, but it is satisfied by $T_{d,n}$ itself, because of Proposition 3.5.1. Thus, the moduli spaces $T_{d,n}$ have a structure of $\mathbb{F}_1$-constructible sets. □

7. Morphisms and operad structure

We show that the constructions of torifications described in the previous section are compatible with the operad structures.

Let an operad $\mathscr{P}$, in the symmetric monoidal category of varieties over $\mathbb{Z}$ with Cartesian product, be given. Its descent data to $\mathbb{F}_1$ consist of affine torifications such that the composition operations

$$\mathscr{P}(n) \times \mathscr{P}(m_1) \times \cdots \times \mathscr{P}(m_n) \longrightarrow \mathscr{P}(m_1 + \cdots + m_n)$$

and the structure actions of symmetric groups are morphisms of affinely torified varieties. This is a "tourist class" description of [26]. A more systematic treatment requires the explicit introduction of a category of labeled graphs as in [3].

7.1. Categories of trees and operads. We consider a category Γ, whose objects are finite disjoint unions of oriented rooted trees. The morphisms are generated by edge contractions and graftings. The grafting of an oriented tree τ to another oriented tree σ is realized by the morphism $h\colon \tau \amalg \sigma \to \tau\#_{v_0,w}\sigma$, where the involution j_h matches the outgoing tail of the root vertex v_0 of τ with an ingoing tail of a vertex w of σ. The edge contractions are given by morphisms $h_e\colon \tau \to \tau/e$, where the edge e is a j_τ-orbit $e = \{f, f'\}$ of flags $f, f' \in F_\tau$, such that $F_\tau \setminus h_e^F(F_{\tau/e}) = \{f, f'\}$, the map $h_{e,F}^{-1}\colon h_e^F(F_{\tau/e}) \to F_{\tau/e}$ is the identity, and $h_{e,V}\colon V_\tau \to V_{\tau/e}$ maps $\partial_\tau(f)$ and $\partial_\tau(f')$ to the same vertex in τ/e.

It is shown in [26, Section IV.2] that the datum of an operad is equivalent to a monoidal functor $\mathscr{M}$ from a category of trees (forests) with the symmetric monoidal structure given by disjoint union and morphisms generated by graftings and edge contractions, to a symmetric monoidal category $(\mathscr{C}, \otimes)$, with the condition that

$$\mathscr{M}(\tau) = \bigotimes_{v \in V_\tau} \mathscr{M}(\tau_v),$$

where τ_v is the star of the vertex v, see [26, Proposition IV.2.4.1]. The operad composition is identified with the image $\mathscr{M}(\psi)$ of the morphism ψ that assigns to a disjoint union of corollas $\tau \amalg \tau_1 \cdots \amalg \tau_n$ the corolla obtained by first grafting the outgoing tails of the component τ_k to the kth ingoing tail of τ and then contracting all the edges.

7.2. Operad morphisms of $\overline{M}_{0,n}$. We now consider the composition maps that give the operad structure of the moduli spaces $\overline{M}_{0,n}$ and see that these are also compatible with the structure of $\mathbb{F}_1$-constructible sets described above.

Theorem 7.2.1. *Let $\mathscr{M}(n) = \overline{M}_{0,n+1}$. The composition morphisms of the operad*

$$\mathscr{M}(n) \times \mathscr{M}(m_1) \times \cdots \times \mathscr{M}(m_n) \longrightarrow \mathscr{M}(m_1 + \cdots + m_n)$$

are strong morphisms of constructibly torified spaces, with respect to the constructible torifications of Theorem 6.1.1. *Thus, the operad $\mathscr{M}(n)$ descends to an operad of $\mathbb{F}_1$-constructible sets, in the category $\mathscr{CT}^s$ of Proposition* 5.1.1.

Proof. The constructible torification of $\overline{M}_{0,n}$, obtained as in Theorem 6.1.1, is built out of constructible torifications of the open strata $M_{0,k}$ and their products, so that one has a family of compatible constructible torifications on the open strata $\prod_k M_{0,n_k+1}$ with $\sum_k n_k = n$. This implies that the inclusions of the boundary strata $\prod_k \overline{M}_{0,n_k+1}$ are compatible with the constructible torifications, hence they are strong morphisms of $\mathbb{F}_1$-constructible sets. □

The symmetric group $\mathbf{S}_n$ acts on $\overline{M}_{0,n}$ by permuting the marked points.

Proposition 7.2.2. *The elements of the symmetric group $\mathbf{S}_n$ act on $\overline{M}_{0,n}$ as ordinary morphisms of $\mathbb{F}_1$-constructible sets, that is, morphisms in the category $\mathscr{CT}^o$ of Proposition 5.1.1.*

Proof. The constructible torification of $\overline{M}_{0,n}$ described in Theorem 6.1.1 is obtained from a constructible torification of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the points $0, \infty$ as 0-dimensional tori and by the complement $\mathbb{G}_m \setminus \{1\}$ of a 0-dimensional torus in a 1-dimensional torus as the remaining piece of the decomposition into sets of $\mathscr{C}_{\mathbb{F}_1}$. The action of an element $\sigma \in \mathbf{S}_n$ on $M_{0,n}$ is a permutation of the n marked points and is therefore given by an isomorphism of $M_{0,n}$ that sends this choice of a constructible torification into a different choice, obtained by a different initial choice of constructible torification of $\mathbb{P}^1 \setminus \{\sigma(0), \sigma(1), \sigma(\infty)\}$ and a permutation of the factors in the product $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3}$. Thus, the permutation group $\mathbf{S}_n$ acts on $\overline{M}_{0,n}$ by ordinary (not strong) morphisms of $\mathbb{F}_1$-constructible sets. $\square$

7.3. Operad morphisms of $T_{d,n}$. The varieties $T_{d,n}$ have natural morphisms defining an operad structure that generalizes the operad of $\overline{M}_{0,n}$. We use here, for convenience, the notation $T_{d,S}$, with S the set of marked points, with $\#S = n$.

Theorem 7.3.1. *For each fixed $d \geq 1$, there are morphisms of the following form, which determine an operad $\mathscr{T}_d$:*

(1) *isomorphisms: $T_{d,S} \xrightarrow{\sim} T_{d,S'}$ for $S' \xrightarrow{\sim} S$, functorial with respect to the bijections of labeling sets.*

(2) *embeddings: $T_{d,S'} \times T_{d,S \setminus S' \cup \{\star\}} \hookrightarrow T_{d,S}$, for $S' \subset S$ with $\#S' \geq 2$.*

(3) *forgetful morphisms: $T_{d,S} \to T_{d,S'}$ for $S' \subset S$ with $\#S' \geq 2$.*

These morphisms satisfy the standard identities.

Proof. The existence of morphisms of the form (1) is clear by construction. The cases (2) and (3) follow from the boundary stratification of these varieties constructed in [5, Theorem 3.3.1]. In fact, the boundary of a variety $T_{d,S}$ is given by smooth normal crossings divisors: given any proper subset $S' \subset S$, there is a nonsingular divisor $T_{d,S}(S') \subset T_{d,S}$. These divisors meet transversely and the only non-empty intersections $T_{d,S}(S_1) \cap \cdots \cap T_{d,S}(S_r)$ occur when the sets S_k are nested (each pair is either disjoint or one is a subset of the other). The divisors satisfy $T_{d,S}(S') \simeq T_{d,S'} \times T_{d,S \setminus S' \cup \{\star\}}$. This gives the morphisms (2) coming from the inclusion of the strata. In terms of morphisms of oriented rooted trees, these correspond to the morphisms that graft the outgoing tail of the first tree with the set of incoming tails identified with S' to the incoming tail marked by $\star$ in the second tree. The forgetful morphisms (3) come from the construction of $T_{d,S \cup \{\star\}}$ from $T_{d,S}$ via a sequence of iterated blowups, as in [5, Theorem 3.3.1]. The composition of the projections of this sequence of blowups gives the forgetful morphism $T_{d,S \cup \{\star\}} \to T_{d,S}$. In terms of rooted trees of projective spaces, these correspond to forgetting some of the marked points and contracting the resulting unstable components.

Using the functorial characterization of operads which can be found in [26, Proposition IV.2.4.1], let $(\mathscr{C}, \otimes)$ be the symmetric monoidal category of algebraic varieties with the Cartesian product, and let $(\Gamma, \amalg)$ be the category of oriented rooted forests with disjoint union. The embeddings of the strata determine the morphisms $\mathscr{T}_d(\psi)$, where ψ is the morphism of oriented rooted trees that assigns to a disjoint union of oriented corollas $\tau \amalg \tau_1 \cdots \amalg \tau_n$, where each corolla has only one outgoing tail, the corolla obtained by first grafting the outgoing tails of the component τ_k to the kth ingoing tail of τ and then contracting all the edges. This assignment determines the operad composition operations

$$T_{d,S} \times T_{d,S_1} \times \cdots \times T_{d,S_n} \longrightarrow T_{d,S_1 \cup \cdots \cup S_n},$$

where $n = \#S$ is the number of incoming tails of the trees of projective spaces parametrized by $T_{d,S}$. $\square$

Theorem 7.3.2. *The operad composition operations*

$$T_{d,k} \times T_{d,n_1} \times \cdots \times T_{d,n_k} \longrightarrow T_{d,n_1+\cdots+n_k} \tag{13}$$

are strong morphisms of constructible torifications, hence they define strong morphisms of $\mathbb{F}_1$-constructible sets, that is, morphisms in the category $\mathscr{C}\mathscr{T}^s$ of Proposition 5.1.1.

Proof. We consider the structure of $\mathbb{F}_1$-constructible sets on the moduli spaces $T_{d,n}$ given by the constructible torification obtained as in Theorem 6.2.2. Since this is built as a collection of compatible constructible torifications on all the boundary strata of $T_{d,n}$, we see that the operad composition operations (13), which are obtained from the morphisms of type (2) of Theorem 7.3.1, are inclusions of boundary strata, hence compatible with the constructible torification. $\square$

The result above accounts for the morphisms of type (2) in Theorem 7.3.1. The morphisms of type (1) and (3) also determine morphisms of constructible torifications.

Proposition 7.3.3. *Morphisms of type* (1) *in Theorem* 7.3.1 *are ordinary* (*not strong*) *morphisms of $\mathbb{F}_1$-constructible sets. Morphisms of type* (3) *of Theorem* 7.3.1 *are strong morphisms of $\mathbb{F}_1$-constructible sets, that is, morphisms in the category $\mathscr{C}\mathscr{T}^s$ of Proposition* 5.1.1.

Proof. The case of morphisms of type (1) is analogous to the case of morphisms permuting the marked points of $\overline{M}_{0,n}$, and for the same reason they are ordinary (not strong) $\mathbb{F}_1$-morphisms. Morphisms of type (3) are the forgetful morphisms $T_{d,n+1} \to T_{d,n}$ that forget one of the marked points and contract the unstable components. The restrictions of these projection maps to the open strata $TH_{d,n}$ and $\prod_i TH_{d,n_i}$, with $\sum_i n_i = n$ and $n_i \geq 2$, are given by projections $(\mathbb{A}^d \setminus \{\mathbf{0},\mathbf{1}\})^{n_j-2} \setminus \Delta \to (\mathbb{A}^d \setminus \{\mathbf{0},\mathbf{1}\})^{n_j-3}$ on one of the factors with $n_j > 2$. By construction of the constructible torifications on the $T_{d,n}$ given in Theorem 6.2.2, these projections are morphisms of $\mathbb{F}_1$-constructible sets. $\square$

8. Moduli spaces and wonderful compactifications

Another approach to defining $\mathbb{F}_1$-structures on the operads of the moduli spaces $\overline{M}_{0,n}$ and of the $T_{d,n}$ is based on the construction of the moduli spaces $T_{d,n}$ as iterated blowups and their relation to the Fulton–MacPherson compactifications.

8.1. Moduli spaces $T_{d,n}$ and Fulton–MacPherson spaces. Let us denote by $X[S]$ the Fulton–MacPherson space. We describe its construction in terms of iterated blowups, following the general construction for graph configuration spaces used in [17], [18] and in [4], in the special case of the complete graph. One starts with the product X^S of $n = \#S$ copies of X and considers all diagonals $\Delta_{S'} \subset X^S$ for all subsets $S' \subseteq S$, given by $\Delta_{S'} = \{x \in X^S \mid x_i = x_j, \forall\, i, j \in S'\}$. Upon identifying the subset S' with the set of vertices of a subgraph $\Gamma_{S'} \subset \Gamma_S$, where Γ_S is the complete graph on the set S of vertices, the diagonal $\Delta_{S'}$ is identified with a product $X^{V_{\Gamma_S/\Gamma_{S'}}}$ where the quotient graph $\Gamma_S/\Gamma_{S'}$ is obtained by identifying all of $\Gamma_{S'}$ with a single vertex. Consider the set $\mathscr{G}_S$ of all subgraphs $\Gamma_{S'}$ that are biconnected (that is, that cannot be disconnected by removing the star of any one vertex) and choose an ordering $\mathscr{G}_S = \{\Gamma_{S'_1}, \dots, \Gamma_{S'_N}\}$ such that if $S_i \supseteq S_j$, then the indices are ordered by $i \leq j$. By *dominant transform* of a subvariety under a blowup one means the proper transform if the variety is not contained in the blowup locus, and the inverse image otherwise (see [17, Definition 2.7]). It was shown in [17, Theorem 1.3 and Proposition 2.13] (see also [4, Proposition 2]) that the sequence of blowups $Y^{(k)}$ with $Y^{(0)} = X^S$ and $Y^{(k+1)}$ obtained by blowing up $Y^{(k)}$ along the dominant transform of $\Delta_{S'_k}$, gives $Y^{(N)} = X[S]$, the Fulton–MacPherson compactification. Let $D(S')$ be the divisors on $X[S]$ obtained as iterated dominant transforms of the diagonals $\Delta_{S'}$, for $\Gamma_{S'}$ in $\mathscr{G}_S$. By [17, Theorem 1.2] and [4, Proposition 4], the intersections $D(S'_{k_1}) \cap \cdots \cap D(S'_{k_r})$ are non-empty if and only if the collection of graphs $\mathscr{N} = \{\Gamma_{S'_{k_1}}, \dots, \Gamma_{S'_{k_r}}\}$ forms a $\mathscr{G}_S$*-nest*, that is, it is a set of biconnected subgraphs of type $\Gamma_{S'}$ such that any two subgraphs are either disjoint, or they intersect at a single vertex, or one is contained as subgraph in the other (see [17, Section 4.3] and [4, Proposition 3]). The varieties $T_{d,S}$ can be identified with the fibers of the projection $\pi\colon D(S) \to X \simeq \Delta_S \subset X^S$, for any smooth variety X of dimension d. In particular, we can use $X = \mathbb{P}^d$.

8.2. Blowup of diagonals and torifications. The wonderful compactification $X[n]$ for $X = \mathbb{P}^d$ is obtained as described above, as an iterated sequence of blowups of the dominant transforms of the diagonals that correspond to all the biconnected subgraphs of the complete graph on n vertices.

Lemma 8.2.1. *The choice of a geometric torification of projective spaces compatible with their cell decomposition determines a geometric torification of the blowup* $\mathrm{Bl}_\Delta\big((\mathbb{P}^d)^n\big)$ *of a diagonal* Δ *inside the product* $(\mathbb{P}^d)^n$*. The morphism* $\pi\colon \mathrm{Bl}_\Delta\big((\mathbb{P}^d)^n\big) \to (\mathbb{P}^d)^n$ *is a weak* $\mathbb{F}_1$*-morphism, with respect to the product torification on the base.*

Proof. The diagonal itself can be identified with a product of copies of $\mathbb{P}^d$, so it has a geometric torification induced by the choice of torification of $\mathbb{P}^d$. The exceptional divisor of the blowup then also has a geometric torification, determined by the torification of Δ and a torification of $\mathbb{P}^{\mathrm{codim}(\Delta)}$, as in Proposition 4.2.2. Thus, we need to check that the complement $(\mathbb{P}^d)^n \setminus \Delta$ also has a geometric torification. It suffices to show this for the deepest diagonal, as in other cases one can split off a factor that can be torified as a product of copies of $\mathbb{P}^d$. Consider the cell decomposition $\mathbb{P}^d = \bigcup_{i=0}^d \mathbb{A}^k$ and the induced cell decomposition of $(\mathbb{P}^d)^n$. The deepest diagonal meets the cells $(\mathbb{A}^k)^n$ of this decomposition. Thus, to construct a geometric torification of $(\mathbb{P}^d)^n \setminus \Delta$ we can use the product torification on all the cells $\prod_{i=1}^n \mathbb{A}^{k_i}$ with not all the k_i the same, and construct a torification of the complements of the diagonal in the affine spaces $(\mathbb{A}^k)^n \setminus \Delta_k$, with $\Delta_k = \Delta \cap (\mathbb{A}^k)^n$. This can be achieved by a change of variables from the standard torification of the product of affine spaces. Thus, if we consider the product torification on $(\mathbb{P}^d)^n$, the morphism $\pi\colon \mathrm{Bl}_\Delta\big((\mathbb{P}^d)^n\big) \to (\mathbb{P}^d)^n$ is compatible with torifications only in the weak sense: there is a decomposition (the cell decomposition) of the variety such that there are isomorphisms on the pieces of the decomposition which perform the change of torification that makes the morphism torified, but these isomorphisms do not extend globally to the variety. □

Thus, in a construction of geometric or constructible torifications on the compactifications $\mathbb{P}^d[n]$ based on iterated blowups, as in [1] the maps $\pi\colon \mathbb{P}^d[n] \to (\mathbb{P}^d)^n$ will only be weak $\mathbb{F}_1$-morphisms, that is, morphisms in the category $\mathscr{CT}^w$ of Proposition 5.1.1.

9. Blueprint structures

As recalled in Remark 2.2.4, one can also consider the less restrictive approach to $\mathbb{F}_1$-structures based on blueprints as in [21]. Here we make explicit a blueprint structure of $\overline{M}_{0,n}$ based upon explicit equations for $\overline{M}_{0,n}$, as in [12], [16]. We also describe a blueprint structure on the genus-zero boundary $\overline{M}^0_{g,n+1}$ of the higher-genus moduli spaces, using a crossed product construction.

Recall that a *blueprint* $\mathscr{A} /\!\!/ \mathscr{R}$ is constructed by considering a commutative multiplicative monoid $\mathscr{A}$ and the associated semiring $\mathbb{N}[\mathscr{A}]$, together with a set of relations $\mathscr{R} \subset \mathbb{N}[\mathscr{A}] \times \mathbb{N}[\mathscr{A}]$, written as relations $\sum a_i \equiv \sum b_j$, for $(\sum a_i, \sum b_j) \in \mathscr{R}$.

Much more details on blueprints can be found in Lorscheid's contribution to this volume.

9.1. $\overline{M}_{0,n}$ and toric varieties. In [12], [13] and [29], one considers a simplicial complex $\tilde{\Delta}$ with the set of vertices $\mathscr{I} = \big\{I \subset \{1, \dots n\}, 1 \in I, \#I \geq 2, \#I^c \geq 2\big\}$ and with simplexes $\sigma \subset \tilde{\Delta}$ if for all I and J in σ either $I \subseteq J$ or $J \subseteq I$ or $I \cup J = \{1, \dots, n\}$. The collection of cones associated to the simplexes σ in $\tilde{\Delta}$ determines a polyhedral fan Δ in $\mathbb{R}^{\binom{n}{2}-1}$, which also arises in tropical geometry as the space of phylogenetic trees [28]. The associated toric variety X_Δ is smooth, though not complete. The moduli space $\overline{M}_{0,n}$ embeds in X_Δ and it intersects the torus T of X_Δ in $M_{0,n}$. The boundary strata of $\overline{M}_{0,n}$ are pullbacks of torus-invariant loci in X_Δ (see [13, Section 6] and [12, Section 5]).

9.2. A blueprint structure on $\overline{M}_{0,n}$. The construction of the toric variety X_Δ in [12], [13], and [29] with the embedding $\overline{M}_{0,n} \hookrightarrow X_\Delta$, relies on an earlier result of Kapranov realizing $\overline{M}_{0,n}$ as a quotient of a Grassmannian. More precisely, in [14], Kapranov showed that the quotient $\mathtt{Grass}^0(2,n)/T$ of the open cell $\mathtt{Grass}^0(2,n)$ of points with non-vanishing Plücker coordinates in the Grassmannian $\mathtt{Grass}(2,n)$, by the action of an $(n-1)$-dimensional torus T, is the moduli space $M_{0,n}$, and its compactification $\overline{M}_{0,n}$ is obtained as the (Chow or Hilbert) quotient of $\mathtt{Grass}(2,n)$ by the action of T.

From the blueprint point of view on $\mathbb{F}_1$-geometry, observe that the Plücker embedding of the Grassmannian $\mathtt{Grass}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$, used to obtain $\overline{M}_{0,n}$ in this way, also furnishes $\mathtt{Grass}(2,n)$ with an $\mathbb{F}_1$-structure as *blueprint* in the sense of [20], [21] (but not as affinely torified varieties), where the blueprint structure (see [21] and [20, Section 5]) is defined by the congruence $\mathscr{R}$ generated by the Plücker relations $x_{ij}x_{kl} + x_{il}x_{jk} = x_{ik}x_{jl}$ for $1 \le i < j < k < l \le n$.

One can use the Plücker coordinates, together with the toric variety construction of [12], [13] and [29], to obtain explicit equations for $\overline{M}_{0,n}$ in the Cox ring of the toric variety X_Δ, see [12, Theorem 1.2] and [16]. This can be used to give a blueprint structure on $\overline{M}_{0,n}$.

Theorem 9.2.1. *The moduli spaces $\overline{M}_{0,n}$ have a blueprint structure*

$$\mathscr{O}_{\mathbb{F}_1}(\overline{M}_{0,n}) = \mathscr{A} /\!\!/ \mathscr{R},$$

where, denoting by $\mathbb{Q}[x_I : I \in \mathscr{I}]$ the Cox ring of X_Δ, $\mathscr{A}$ is the monoid

$$\mathscr{A} = \mathbb{F}_1[x_I : I \in \mathscr{I}] := \Big\{ \prod_I x_I^{n_I} \Big\}_{n_I \ge 0},$$

and the blueprint relations are given by $\mathscr{R} = \mathscr{S}_f^{-1}\mathscr{R}' \cap \mathscr{A}$, where we denote by $\mathscr{R}'$ the set

$$\mathscr{R}' = \Big\{ \prod_{ij\in I, kl\notin I} x_I + \prod_{il\in I, jk\notin I} x_I \equiv \prod_{ik\in I, jl\notin I} x_I : 1 \le i < j < k < l \le n \Big\},$$

and by $\mathscr{S}_f^{-1}\mathscr{R}'$ the localization of $\mathscr{R}'$ with respect to the submonoid generated by the element $f = \prod_I x_I$.

Proof. In [12, Proposition 2.1] one finds a general method for producing explicit equations for quotients of subvarieties of a torus by the action of a subtorus, and [12, Theorem 3.2] uses this result to obtain explicit equations for Chow and Hilbert quotients of T^d-equivariant subschemes of projective spaces $\mathbb{P}^m$. Then, [12, Theorem 6.3] obtains explicit equations for $\overline{M}_{0,n}$ inside the toric variety X_Δ starting with the Plücker relations on the Grassmannian $\mathtt{Grass}(2,n)$ and the quotient description of $\overline{M}_{0,n}$ obtained in [14]. More precisely, the equations for $\overline{M}_{0,n}$ are obtained by homogenizing the Plücker relations with respect to the grading in the Cox ring of X_Δ, and then saturating by the product of the variables in the Cox ring. With the notation $\mathfrak{I} : \mathfrak{J}^\infty$ for the saturation of an ideal $\mathfrak{I}$ by $\mathfrak{J}$, the equations for $\overline{M}_{0,n}$ are given by (see [12, Theorem 6.3])

$$\Big\langle \prod_{ij\in I, kl\notin I} x_I - \prod_{ik\in I, jl\notin I} x_I + \prod_{il\in I, jk\notin I} x_I \Big\rangle : \Big(\prod_I x_I\Big)^\infty,$$

where $\mathbb{Q}[x_I, : I \in \mathscr{I}]$ is the Cox ring of X_Δ, and where i, j, k, l satisfy $1 \leq i < j < k < l \leq n$.

In general, let A be a polynomial ring and $\mathfrak{I}$ an ideal, and let $\mathfrak{J} = (f)$ be the ideal generated by an element f. Then the saturation $\mathfrak{I} : \mathfrak{J}^\infty$ is $\mathfrak{I}_f \cap A$, where $\mathfrak{I}_f$ is the localization of $\mathfrak{I}$ at f. Thus, we can write the ideal of $\overline{M}_{0,n}$ in terms of localizations.

As in [20, Section 4], we can consider the blueprint $\mathscr{B}' = \mathscr{A} /\!\!/ \mathscr{R}'$, with the monoid $\mathscr{A} = \mathbb{F}_1[x_I : I \in \mathscr{I}]$ and the blueprint relations $\mathscr{R}'$ given in the statement. As shown in [21, Section 1.13], blueprints admit localizations with respect to submonoids of $\mathscr{A}$. Thus, given the element $f = \prod_I x_I$, and letting $\mathscr{S}_f$ be the submonoid of $\mathscr{A}$ generated by f, we can consider the localization

$$\mathscr{S}_f^{-1}\mathscr{R} = \left\{ \prod_{ij\in I, kl\notin I} \frac{x_I}{f^{a_I}} + \prod_{il\in I, jk\notin I} \frac{x_I}{f^{a_I}} \equiv \prod_{ik\in I, jl\notin I} \frac{x_I}{f^{a_I}} : 1 \leq i < j < k < l \leq n \right\},$$

where the localized blueprint relation $\mathscr{S}_f^{-1}\mathscr{R}'$ lives in the localization $\mathscr{S}_f^{-1}\mathscr{A} \subset \mathscr{A} \times \mathscr{S}_f$, given by the set of equivalence classes (denoted a/f^k) of elements (a, f^k) with the relation $(a, f^k) \sim (b, f^\ell)$ when $f^{k+m}b = f^{\ell+m}a$ for some m. The blueprint relations $\mathscr{R} = \mathscr{S}_f^{-1}\mathscr{R}' \cap \mathscr{A}$ then give the blueprint structure of $\overline{M}_{0,n}$. □

9.3. Remarks on higher genera. The moduli spaces $M_{g,n}$ of stable curves of higher genus with marked points have Deligne–Mumford compactifications $\overline{M}_{g,n}$, with natural morphisms between them, similar to the genus-zero case: inclusions of boundary strata

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \longrightarrow \overline{M}_{g_1+g_2,n_1+n_2}$$

and forgetting markings (and stabilizing)

$$\overline{M}_{g,n} \longrightarrow \overline{M}_{g,n-1},$$

as well as morphisms arising from gluing two marked points together,

$$\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}.$$

However, $\overline{M}_{g,n}$ are generally only stacks rather than schemes.

One does not expect higher-genus moduli spaces to carry $\mathbb{F}_1$-structures in the approach based on torifications (though they can have blueprint structures). However, one can consider interesting sub-loci of these moduli spaces, like $\overline{M}^0_{g,n}$, parametrizing curves whose irreducible components are all rational. These stacks can be made components of an operad, and at least some covers of them admit a compatible $\mathbb{F}_1$-structure. In order to complete this picture, the basics of DM-stacks theory over $\mathbb{F}_1$ must be developed first.

9.4. Blueprints and the $\overline{M}^0_{g,n}$ strata. The locus $\overline{M}^0_{g,n}$ of rational curves in the higher-genus moduli space $\overline{M}_{g,n}$ can be described, as explained in [11], as the image of a finite map

$$R\colon \overline{M}_{0,2g+n} \longrightarrow \overline{M}_{g,n},$$

obtained in the following way. The locus $\overline{M}^0_{g,n}$ is the closure of the locus of irreducible g-nodal curves. These curves have a normalization given by a smooth rational curve with $2g+n$ marked points. One can then consider the subgroup $G \subset \mathbf{S}_{2g}$ of permutations of these $2g$ additional marked points that commute with the product $(12)(34)\cdots(2g-1\ 2g)$ of g transpositions, so that the normalization of $\overline{M}^0_{g,n}$ can be identified with the quotient $\overline{M}_{0,2g+n}/G$.

We say that a group G acts on a blueprint $\mathscr{A}\,/\!\!/\,\mathscr{R}$ *by automorphisms* if it acts by automorphisms on the monoid $\mathscr{A}$ and the induced diagonal action on $\mathbb{N}[\mathscr{A}]\times\mathbb{N}[\mathscr{A}]$ preserves the set of blueprint relations $\mathscr{R}$.

Lemma 9.4.1. *The action of G on $\overline{M}_{0,2g+n}$ induces an action by automorphisms on the blueprint $\mathscr{O}_{\mathbb{F}_1}(\overline{M}_{0,2g+n})$.*

Proof. In general, the action of the symmetric group $\mathbf{S}_n$ on $\overline{M}_{0,n}$ by permutation of the marked points induces an action by automorphisms on the commutative monoid $\mathscr{A}=\mathbb{F}_1[x_I : I\in\mathscr{I}]$ described above, by correspondingly permuting the coordinates x_I. This action fixes the element $f=\prod_I x_I$ and preserves the set of blueprint relations $\mathscr{R}$, because it corresponds to the action on the set of Plücker relations by permuting matrix columns. Thus, the subgroup $G\subset\mathbf{S}_{2g}\subset\mathbf{S}_{2g+n}$ also acts by automorphisms on the monoid $\mathscr{A}$ of $\overline{M}_{0,2g+n}$ preserving the blueprint relations, hence as automorphisms of $\mathscr{O}_{\mathbb{F}_1}(\overline{M}_{0,2g+n})$. □

In order to obtain $\mathbb{F}_1$-data for the quotient $\overline{M}_{0,2g+n}/G$, we suggest an approach that uses the point of view of noncommutative geometry, replacing the quotient operation by a crossed product by the group of symmetries, at the level of the associated algebraic structure. This point of view suggests introducing a notion of (non-commutative) crossed product blueprints.

Definition 9.4.2. Let $\mathscr{A}\,/\!\!/\,\mathscr{R}$ be a blueprint with $\mathscr{A}$ a commutative multiplicative monoid and $\mathscr{R}$ a set of blueprint relations, and let G be a group of automorphisms of $\mathscr{A}\,/\!\!/\,\mathscr{R}$. The *monoid crossed product* $\mathscr{A}\rtimes G$ is the multiplicative (non-commutative) monoid with elements of the form (a,g) with $a\in\mathscr{A}$ and $g\in G$, and with product $(a,g)(a',g')=(ag(a'),gg')$. The *semiring crossed product* $\mathbb{N}[\mathscr{A}]\rtimes G$ is given by all finite formal sums $\sum(a_i,g_i)$ with $a_i\in\mathscr{A}$ and $g_i\in G$, and with multiplication $(a_i,g_i)(a_j,g_j)=(a_ig_i(a_j),g_ig_j)$. Let

$$\mathscr{R}_G\subset(\mathbb{N}[\mathscr{A}]\rtimes G)\times(\mathbb{N}[\mathscr{A}]\rtimes G)$$

be the set of elements $\big((\sum a_i,g),(\sum b_j,g)\big)$, with $(\sum a_i,\sum b_j)\in\mathscr{R}$ and $g\in G$. The crossed product $(\mathscr{A}\,/\!\!/\,\mathscr{R})\rtimes G$ is defined as the pair $(\mathscr{A}\rtimes G,\mathscr{R}_G)$.

Lemma 9.4.3. *The action of the symmetric group $\mathbf{S}_n$ on the moduli space $\overline{M}_{0,n}$ determines a crossed product blueprint $\mathscr{O}_{\mathbb{F}_1}(\overline{M}_{0,n})\rtimes\mathbf{GL}_n(\mathbb{F}_1)$.*

Proof. This is an immediate consequence of Lemma 9.4.1, Definition 9.4.2, and the identification $\mathbf{S}_n=\mathbf{GL}_n(\mathbb{F}_1)$. □

We can then use this notion of crossed product blueprint to associate $\mathbb{F}_1$-data to the strata $\overline{M}^0_{g,n}$ of the higher-genus moduli spaces $\overline{M}_{g,n}$.

Proposition 9.4.4. *The normalization of* $\overline{M}^0_{g,n}$ *has an associated crossed product blueprint structure* $\mathscr{O}_{\mathbb{F}_1}(\overline{M}_{0,2g+n}) \rtimes G$*, with* $G \subset \mathbf{S}_{2g}$ *the subgroup of permutations that commute with the product of transpositions* $(12)(34)\cdots(2g-1\ 2g)$.

Proof. Again, this follows from Lemma 9.4.1 and from Definition 9.4.2. □

As in noncommutative geometry, the use of crossed product structures is a convenient replacement for the quotient $\overline{M}_{0,2g+n}/G$.

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Index