

## UNIVERSAL Menger COMPACTA AND UNIVERSAL MAPPINGS

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**ABSTRACT.** For any positive integer  $n$  the author constructs a continuous mapping  $f_n: M_n \rightarrow M_n$  of the  $n$ -dimensional Menger compactum onto itself that is universal in the class of mappings between  $n$ -dimensional compacta, i.e., for any continuous mapping  $g: X \rightarrow Y$  between  $n$ -dimensional compacta there exist imbeddings of  $X$  and  $Y$  in  $M_n$  such that the restriction of  $f_n$  to  $X$  is homeomorphic to  $g$ . The mapping  $f_n$  plays the same role in the theory of Menger  $n$ -dimensional manifolds as the projection  $\pi: Q \times Q \rightarrow Q$  plays in the theory of  $Q$ -manifolds ( $Q$  is the Hilbert cube). It can be used to carry over the classical theorems in the theory of  $Q$ -manifolds to the theory of  $M_n$ -manifolds:

**STABILIZATION THEOREM.** For any  $M_n$ -manifold  $X$  and any imbedding of  $X$  in  $M_n$  the space  $f_n^{-1}(X)$  is homeomorphic to  $X$ .

**TRIANGULATION THEOREM.** For any  $M_n$ -manifold  $X$  there exists an  $n$ -dimensional polyhedron  $K$  such that the space  $f_n^{-1}(K)$  is homeomorphic to  $X$  for every imbedding of  $K$  in  $M_n$ .

Bibliography: 20 titles.

A classical result in geometric topology is the Nöbeling-Pontryagin theorem [1], [2], proved in 1931, on the universality of  $(2n + 1)$ -dimensional Euclidean space for  $n$ -dimensional separable metric spaces. Nöbeling's proof consisted in a proof of universality for the subset  $N_n^{2n+1}$  (the so-called  $n$ -dimensional Nöbeling space) of  $(2n + 1)$ -dimensional Euclidean space constructed by him. The Nöbeling spaces provide a noncompact analogue of the Menger spaces  $M_n^{2n+1}$ , constructed by Menger in 1926 [3]; the universality of the latter in the class of  $n$ -dimensional separable metric spaces was proved formally only in 1963 by Bothe [4]. The universality of the compacta  $M_n^{2n+1}$  was actually proved in 1931 by Lefschetz [5]. He constructed compacta  $S_n^{2n+1}$  and proved their universality, and in 1984 the Yugoslav mathematician Bestvina announced at the Banach Seminar in Warsaw a theorem [6] which implies that  $M_n^{2n+1}$  and  $S_n^{2n+1}$  are homeomorphic.

**BESTVINA'S THEOREM.** A compact  $n$ -dimensional absolute extensor in dimension  $n$  that satisfies the condition  $DD^nP$  is homeomorphic to the universal Menger compactum  $M_n^{2n+1}$ .

For  $n = 1$  this theorem was proved by the author [7] with the help of a characterization of the universal Menger curve due to Anderson [8]. Bestvina's theorem implies that all the Menger compacta  $M_n^m$  are mutually homeomorphic for  $m \geq 2n + 1$ , and thus we can denote them simply by  $M_n$ .

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A few words about the concepts in the statement of Bestvina’s theorem. Compact absolute extensors in the class of  $n$ -dimensional spaces are, in view of a theorem of Kuratowski [9], precisely the  $(n - 1)$ -connected and locally  $(n - 1)$ -connected compacta. The  $DD^nP$  condition is the condition of disjoint approximation for mappings of  $n$ -dimensional disks:

( $DD^nP$ ) A space  $X$  satisfies the condition  $DD^nP$  if for any two continuous mappings  $f: D^n \rightarrow X$  and  $g: D^n \rightarrow X$  of  $n$ -dimensional disks and for any  $\varepsilon > 0$  there exist continuous mappings  $f_\varepsilon: D^n \rightarrow X$  and  $g_\varepsilon: D^n \rightarrow X$  that are  $\varepsilon$ -close to  $f$  and  $g$ , respectively, and such that  $f_\varepsilon(D^n) \cap g_\varepsilon(D^n) = \emptyset$ .

The present article is closely connected with the author’s paper [7], which is devoted to a study of  $AE(n)$ -spaces and  $n$ -soft mappings. In a way similar to the way the theory of  $ANE$ -spaces (absolute neighborhood extensors =  $ANE$ ’s) and the theory of  $Q$ -manifolds are closely connected, a connection emerges between the theory of absolute neighborhood extensors in dimension  $n$  ( $ANE(n) = LC^{n-1}$ ) and the theory of Menger manifolds, which is in the construction process now. We illustrate this statement by a table (see below).

	$Q$ -manifolds and A(N)E-compacta	$M_n$ -manifolds and A(N)E( $n$ )-compacta
1.	The Hilbert cube is topologically homogeneous [10].	The compactum $M_n$ is topologically homogeneous [6].
2.	The Toruńczyk criterion [11]: An $ANE$ -compactum with the $DD^\infty P$ condition is a $Q$ -manifold.	The Bestvina criterion [6]: an $n$ -dimensional $ANE(n)$ -compactum with $DD^n P$ condition is an $M_n$ -manifold.
3.	$AE$ -compacta are precisely absolutely invertible images (images under retractions) of the Hilbert cube $Q$ .	$AE(n)$ -compacta are precisely the $n$ -invertible images of the Menger compactum $M_n$ [7].
4.	Every $ANE$ -compactum is a $CE$ -image of a $Q$ -manifold [12].	Every $ANE(n)$ -compactum is a $UV^{n-1}$ image of an $M_n$ -manifold [6].

As is known, in the case of  $Q$ -manifolds assertion 4 can be strengthened to the following: every  $ANE$ -compactum is an absolutely soft image of a  $Q$ -manifold and, what is more (Edwards; see [12]), the product of an  $ANE$ -compactum by the Hilbert cube is a  $Q$ -manifold. In this article we give a theorem about Menger manifolds (Theorem 2) analogous to the first of these results.

Recall ([7], [13]) that a mapping  $f: X \rightarrow Y$  is said to be  $(n, m)$ -soft if for any  $m$ -dimensional space  $Z$ , any  $n$ -dimensional closed subset  $A \subset Z$ , and any continuous mappings  $\psi: Z \rightarrow Y$  and  $\varphi: A \rightarrow X$  there exists a continuous extension  $\bar{\varphi}: Z \rightarrow X$  with  $f \circ \bar{\varphi} = \psi$ . A mapping is said to be  $n$ -invertible if it is  $(-1, n)$ -soft; an  $(n, n)$ -soft mapping is said to be  $n$ -soft, and an  $(\infty, \infty)$ -soft mapping is said to be absolutely soft.

In the present article we construct in §1 universal mappings  $f_n$  and  $g_n$  in the class of all possible continuous mappings of at most  $n$ -dimensional separable metric spaces and in the class of continuous mappings between at most  $n$ -dimensional separable metric spaces. The mapping  $f_n$  carries the Menger compactum  $M_n$  onto the Hilbert cube, and the mapping  $g_n$  carries  $M_n$  onto itself. The universality of the mappings  $f_n: M_n \rightarrow Q$  and  $g_n: M_n \rightarrow M_n$  means that for any continuous mappings of separable metric spaces  $\alpha: X_1 \rightarrow Y_1$  and  $\beta: X_2 \rightarrow Y_2$  with  $\max\{\dim X_1, \dim X_2, \dim Y_2\} \leq n$ , there exist imbeddings  $i_1: X_1 \hookrightarrow M_n$ ,  $j: Y_1 \hookrightarrow Q$ ,  $i_2: X_2 \hookrightarrow M_n$ , and  $j_2: Y_2 \hookrightarrow M_n$  such that the diagrams obtained commute. The

mappings  $f_n$  and  $g_n$  are open, and all their fibers are homeomorphic to  $M_n$ . For  $n = 1$  a similar mapping was constructed by Anderson in [14], but it was not proved to be universal. The mappings  $f_n$  and  $g_n$  satisfy the additional conditions of softness (Theorem 1 and Theorem 3), along with a parametric version of  $DD^nP$ . In my opinion, the taking of the complete inverse image under  $g_n$  is the analogue in the theory of Menger manifolds of multiplication by the Hilbert cube in the theory of  $Q$ -manifolds. Theorems 6 and 7 serve as a confirmation of this. The question arises of the uniqueness of the mappings  $f_n$  and  $g_n$ , and a positive answer to this question would give new impetus to the development of the theory of Menger manifolds.

In the nonmetrizable case, the theory of  $AE(n)$ -compact Hausdorff spaces is not completely analogous to the theory of  $AE(n)$ -compacta (by a compactum we mean an everywhere metrizable compact Hausdorff space). For example, there does not exist an  $n$ -dimensional  $AE(n)$ -compact Hausdorff space of uncountable weight [15], and the classes  $AE(n + 1)$  and  $AE(n, \infty)$  do not coincide, as observed in the metric case. The definition of  $AE(n, m)$ -spaces is due to Hoffmann [16]: a compact Hausdorff space  $X$  belongs to the class  $AE(n, m)$  if a constant mapping of it is  $(n, m)$ -soft. Theorem 10 in §2 gives a characterization of  $AE(n, m)$ -compact Hausdorff spaces for  $m - n > 1$  as  $n$ -invertible images of the universal compact Hausdorff spaces  $D_m^\tau$ . The latter spaces were introduced in [7] and have many remarkable properties analogous to the properties of the Menger compacta  $M_n$ . In the present article we prove properties of the spaces  $D_m^\tau$  in addition to those in [7], and use them to characterize  $AE(n, m)$ .

It follows from the definition that an  $(n + 1)$ -soft image of an  $AE(n + 1)$ -space is an  $AE(n + 1)$ -space. In the metric case this holds also for an  $n$ -soft mapping, since  $AE(n + 1) = AE(n, n + 1)$ . In the case of compact Hausdorff spaces of arbitrary weight the equality  $AE(n + 1) = AE(n, n + 1)$  has not been proved (it may fail to hold), and the above assertion is not obvious. It is proved in Theorem 13, and it was first proved by Chigogidze for mappings with a metrizable kernel.

### §1. Menger compacta and universal mappings

A mapping  $f: X \rightarrow Y$  is said to be polyhedrally  $n$ -soft if the  $n$ -softness condition holds for pairs  $(Z, A)$ , where  $Z$  is an  $n$ -dimensional polyhedron and  $A$  is a subpolyhedron of it. We say that a mapping  $f: X \rightarrow Y$  satisfies the parametric version of  $DD^nP$  if for any two mappings  $\alpha: B^n \rightarrow X$  and  $\beta: B^n \rightarrow X$  of an  $n$ -dimensional disk and for any  $\epsilon > 0$  there exist mappings  $\alpha_\epsilon: B^n \rightarrow X$  and  $\beta_\epsilon: B^n \rightarrow X$  that are  $\epsilon$ -close to  $\alpha$  and  $\beta$ , respectively, with disjoint ranges and with the property that  $f \circ \alpha = f \circ \alpha_\epsilon$  and  $f \circ \beta = f \circ \beta_\epsilon$ .

**THEOREM 1.** *For any positive integer  $n$  there exists a continuous mapping  $f_n: M_n \rightarrow Q$  of the universal Menger  $n$ -dimensional compactum onto the Hilbert cube having the following properties:*

- 1) *All the fibers of the mapping  $f_n$  are homeomorphic to the Menger compactum  $M_n$ .*
- 2) *The mapping  $f_n$  is  $(n - 2, n)$ -soft.*
- 3) *The mapping  $f_n$  is  $(n - 1)$ -soft.*
- 4) *The mapping  $f_n$  is polyhedrally  $n$ -soft.*
- 5) *The mapping  $f_n$  satisfies the parametric version of  $DD^nP$ .*
- 6) *The mapping  $f_n$  is universal in the class of mappings of at most  $n$ -dimensional separable metric spaces.*

In [7] we constructed an  $(n - 1)$ -soft  $n$ -invertible mapping  $f: M_n \rightarrow Q$ . The mapping  $f_n$  will be constructed by modifying the construction of  $f$ . Let  $E^{n+1}$  be  $(n + 1)$ -dimensional Euclidean space, and let  $B^{n+1} = \{x \in E^{n+1}: |x| \leq 1\}$ . Let  $R_n: B^{n+1} \rightarrow \partial B^{n+1}$  be the following multivalued retraction of the  $(n + 1)$ -dimensional ball  $B^{n+1}$  onto the boundary  $\partial B^{n+1} = S^n$ :  $R_n(x) = \{y \in S^n: (x, y) \geq 4|x|^2 - 3|x|\}$ . This multivalued retraction is notable for the fact that it associates the whole sphere  $S^n$  with every point  $x$  in the ball  $B_0^{n+1} = \{x \in E^{n+1}: |x| \leq 1/2\}$ , and some ball on  $S^n$  with a point in  $\text{Int } B^{n+1} \setminus B_0^{n+1}$ . Obviously,  $R_n$  is a retraction (that is,  $R_n(x) = \{x\}$  for  $x \in S^n$ ), and it is continuous. Let  $\Gamma_{R_n}$  denote the graph of  $R_n$ , and let  $p_n: \Gamma_{R_n} \rightarrow B^{n+1}$  and  $q_n: \Gamma_{R_n} \rightarrow S^n$  denote its projections.

LEMMA 1. *The mapping  $p_n: \Gamma_{R_n} \rightarrow B^{n+1}$  is  $(n - 2, n)$ -soft.*

LEMMA 2. *The mapping  $p_n: \Gamma_{R_n} \rightarrow B^{n+1}$  is polyhedrally  $n$ -soft.*

The proofs of Lemmas 1 and 2 will be given after we construct  $f_n$ .

For any polyhedron  $K$  let  $K^{(n)}$  denote its  $n$ -dimensional skeleton.

LEMMA 3. *For any polyhedron  $K$  and any triangulation of it there exist a polyhedron  $L$  and continuous mappings  $f: L \rightarrow K$  and  $g: L \rightarrow K^{(n)}$  such that  $f$  is polyhedrally  $n$ -soft and  $(n - 2, n)$ -soft, while  $g \circ f^{-1}: K \rightarrow K^{(n)}$  is a multivalued retraction, and  $g \circ f^{-1}(\sigma) \subset \sigma^{(n)}$  for any simplex  $\sigma$  in  $K$ .*

PROOF. Let  $\dim K = m$ . Let  $R^m$  denote the multivalued retraction of  $K$  onto  $K^{(m-1)}$  obtained by fixing on each  $m$ -dimensional simplex in  $K$  a multivalued retraction homeomorphic to  $R_{m-1}: B^m \rightarrow \partial B^m$ . Let  $\Gamma_m$  denote the graph of  $R^m$ , and  $p^m: \Gamma_m \rightarrow K$  and  $q^m: \Gamma_m \rightarrow K^{(m-1)}$  its projections. Lemmas 1 and 2 imply that the projection  $p^m$  is an  $(m - 3, m - 1)$ -soft and polyhedrally  $(m - 1)$ -soft mapping. For any  $i > n$  let  $R^i, \Gamma_i, p^i$ , and  $q^i$  denote the multivalued retraction of  $K^{(i)}$  onto  $K^{(i-1)}$  constructed in the way described above, its graph, and its projections. As a result we obtain the diagram  $\{K^{(i)}, \Gamma_i; p^i, q^i\}_{i > n}$ :

$$\begin{array}{ccccccc}
 \Gamma_m & & \Gamma_{m-1} & & \cdots & & \Gamma_{n+1} \\
 \downarrow p^m & \searrow q^m & \downarrow p^{m-1} & \searrow q^{m-1} & & & \downarrow p^{n+1} & \searrow q^{n+1} \\
 K & & K^{(m-1)} & & K^{(m-2)} & \cdots & K^{(n+1)} & & K^{(n)}.
 \end{array}$$

Let  $L = \varprojlim \{K^{(i)}, \Gamma_i; p^i, q^i\}$  be the inverse limit of the diagram, and let  $f: L \rightarrow K$  and  $g: L \rightarrow K^{(n)}$  be its limit projections. Since all the projections  $p^i$  are  $(n - 2, n)$ -soft and polyhedrally  $n$ -soft, the projection  $f$  has the same properties. We show this for  $(n - 2, n)$ -softness. Suppose that  $\dim Z = n$ ,  $\dim A = n - 2$ ,  $A$  is closed in  $Z$ , and  $\psi: Z \rightarrow K$  and  $\varphi: A \rightarrow L$  are given mappings with  $f \circ \varphi = \psi|_A$ . The last point means that collections of mappings  $\varphi_i: A \rightarrow K^{(i)}$  and  $\varphi^i: A \rightarrow \Gamma_i$  are given such that all the resulting diagrams commute and  $\varphi_m = \psi|_A$ . To verify that  $f$  is  $(n - 2, n)$ -soft we must construct the collections of extensions  $\bar{\varphi}_i$  and  $\bar{\varphi}^i$  in such a way that all newly formed diagrams commute; that is, in such a way that  $p^i \circ \bar{\varphi}^i = \varphi_i$ ,  $q^i \circ \bar{\varphi}^i = \bar{\varphi}_{i-1}$ , and  $\bar{\varphi}_m = \psi$ . The mapping  $\bar{\varphi}^m$  exists because the projections  $p^m$  are  $(n - 2, n)$ -soft, and  $\bar{\varphi}_{m-1}$  is set equal to  $q^m \circ \bar{\varphi}^m$ . In view of the  $(n - 2, n)$ -softness of  $p^{m-1}$  we can construct the mapping  $\bar{\varphi}^{m-1}$ , and so on, up to  $\bar{\varphi}^{n+1}$  and  $\bar{\varphi}_n$ .

Obviously,  $g \circ f^{-1} = q^{n+1} \circ (p^{n+1})^{-1} \circ \dots \circ q^m \circ (p^m)^{-1}$ , and since  $q^i \circ (p^i)^{-1} = R^i$ , the composition  $g \circ f^{-1}$  is a multivalued retraction of the polyhedron  $K$  onto  $K^{(n)}$ , and  $g \circ f^{-1}(\sigma) \subset \sigma^{(n)}$  for any simplex  $\sigma$  in  $K$ .

For the constructed space  $L$  to be a polyhedron it is necessary to exhibit some subtlety in constructing the multivalued retractions  $R^i$ . Everywhere in what follows we need only that  $L \in \text{ANE}$  (see [7] for a proof).

*Construction of  $f_n$ .* We construct  $f_n$  as the limit projection of an inverse spectrum  $S = \{N_i, \alpha^{i+1}\}$  of  $Q$ -manifolds, where  $N_1 = Q$ . Suppose that the Hilbert cube  $Q$  is represented as a countable product  $\prod_1^\infty Q_i$  of Hilbert cubes, and let each product  $\prod_1^k Q_i$  be imbedded in  $Q$  as the subproduct  $\prod_1^m Q_i \times \prod_{m+1}^\infty \{a_j\}$ , where  $a_i$  is a particular point in  $Q_i$ . Let  $\rho_i$  be a metric on  $Q_i$  bounded by 1, and let  $\rho = \sum_1^\infty \rho_i/2^i$  be the metric on  $Q$ . We construct each manifold  $N_k$  to lie in the product  $\prod_1^k Q_i$  in such a way that the projections  $\alpha_k^{k+1}$  are restrictions of the projections

$$\pi_k^{k+1}: \left( \prod_{i=1}^k Q_i \right) \times Q_{k+1} \rightarrow \prod_{i=1}^k Q_i.$$

In this case the limit  $X = \varprojlim S$  of the spectrum is naturally imbedded in  $Q$ .

We construct the spectrum  $S$  by induction. Suppose that the  $Q$ -manifold  $N_i \subset \prod_1^i Q_i \subset Q$  has been constructed. By Chapman's theorem [12] the  $Q$ -manifold  $N_j$  is homeomorphic to the product of the Hilbert cube  $Q^j$  and the polyhedron  $K_j$ . By trivial arguments,  $K_j$  can be assumed to be such that the projection  $\omega_j: Q^j \times K_j \rightarrow K_j$ , in composition with a certain section  $s_j: K_j \rightarrow Q^j \times K_j$ , is a  $(1/2^j)$ -shift; that is,  $\rho(x, s_j \circ \omega_j(x)) < 1/2^j$  for any  $x \in Q^j \times K_j$ . Suppose that  $K_j$  has a triangulation such that  $\text{diam } s_j(\sigma) < 1/2^j$  for any simplex  $\sigma$ . By Lemma 3, there exist an ANE-compactum  $X_{j+1}$  and mappings  $f_{j+1}: X_{j+1} \rightarrow K_j$  and  $g_{j+1}: X_{j+1} \rightarrow K_j^{(n)}$  such that  $f_{j+1}$  is  $(n-2, n)$ -soft and polyhedrally  $n$ -soft, and  $g_{j+1} \circ f_{j+1}^{-1}(\sigma) \subset \sigma^{(n)}$  for any simplex  $\sigma$  in  $K_j$ . We define  $N_{j+1}$  as the product of the space  $X_{j+1}$  and the Hilbert cube  $Q^{j+1}$ ; this product is a  $Q$ -manifold in view of a theorem of Edwards (see [12]). We split the Hilbert cube  $Q^{j+1}$  into a product of the two cubes  $Q_1^{j+1}$  and  $Q^j$ , and denote by  $\eta_{j+1}$  the projection of the product  $X_{j+1} \times Q_1^{j+1} \times Q^j$  onto  $X_{j+1} \times Q^j$ . We define the mapping  $\beta_j^{j+1}: N_{j+1} \rightarrow N_j$  to be the composition  $(f_{j+1} \times \text{id}_{Q^j}) \circ \eta_{j+1}$ . Let  $\lambda: N_{j+1} \hookrightarrow Q_{j+1}$  be the imbedding; then define the imbedding of  $N_{j+1}$  into  $\prod_1^{j+1} Q_i$  as the diagonal product of  $\beta_j^{j+1}$  and  $\lambda$ . Note that the projection  $\alpha_j^{j+1} = \pi_j^{j+1}|_{N_{j+1}}$  is an  $(n-2, n)$ -soft and polyhedrally  $n$ -soft mapping, being the composition of such a mapping and an absolutely soft mapping.

Thus, the spectrum  $S$  has been constructed. Denote by  $\alpha_i$  the limit projection in the spectrum  $S$ . Let  $\pi: Q_1 \rightarrow Q$  be the trivial bundle with fiber the Hilbert cube. We define  $f_n = \pi \circ \alpha_1$ .

**LEMMA 4.** *The mapping  $f_n: X \rightarrow Q$  is  $(n-2, n)$ -soft and polyhedrally  $n$ -soft.*

The lemma is easily proved from the definition and the necessary softness of the projections  $\alpha_i^{i+1}$ .

**LEMMA 5.** *The dimension of the compactum  $X = \varprojlim S$  is equal to  $n$ .*

**PROOF.** We prove the inequality  $\dim X \leq n$ ; the easier inequality  $\dim X \geq n$  follows, for example, from Lemma 6 below. It suffices to construct for any  $j$  a  $(3/2^j)$ -shift of  $X$

into an  $n$ -dimensional polyhedron. Consider the mapping

$$s_j \circ \omega_j \circ (g_{j+1} \times \text{id}_{Q^j}) \circ \eta_{j+1} \circ \alpha_{j+1}: X \rightarrow K_j^{(n)}.$$

It is the desired  $(3/2^j)$ -shift.

LEMMA 6. *The mapping  $f_n$  satisfies the parametric version of  $DD^nP$ .*

PROOF. Let  $f: B^n \rightarrow X$  and  $g: B^n \rightarrow X$  be given mappings, and let  $\varepsilon > 0$ . There exist an  $i \in \mathbb{N}$  and a  $\delta > 0$  such that the inverse image  $\alpha_i^{-1}(F)$  of every set  $F$  of diameter less than  $\delta$  is less than  $\varepsilon$ . Since  $\eta_{i+1}$  is a trivial bundle with fiber the Hilbert cube, it follows that  $\eta_{i+1}$  satisfies the parametric version of  $DD^nP$ . Consequently, there exist mappings  $f_\delta: B^n \rightarrow N_{i+1}$  and  $g_\delta: B^n \rightarrow N_{i+1}$  with disjoint ranges that are  $\delta$ -close to the mappings  $\alpha_{i+1} \circ f$  and  $\alpha_{i+1} \circ g$ , respectively, with  $\eta_{i+1} \circ f_\delta = \eta_{i+1} \circ \alpha_{i+1} \circ f$  and  $\eta_{i+1} \circ g_\delta = \eta_{i+1} \circ \alpha_{i+1} \circ g$ . Since the projection  $\alpha_{i+1}$  is  $(n-2, n)$ -soft, there exist mappings  $f_\varepsilon: B^n \rightarrow X$  and  $g_\varepsilon: B^n \rightarrow X$  such that  $\alpha_{i+1} \circ f_\varepsilon = f_\delta$  and  $\alpha_{i+1} \circ g_\varepsilon = g_\delta$ . Obviously,  $f_\varepsilon$  and  $f$  as well as  $g_\varepsilon$  and  $g$  are  $\varepsilon$ -close, and  $f_\varepsilon(B^n) \cap g_\varepsilon(B^n) = \emptyset$ , while  $f_n \circ f = f_n \circ f_\varepsilon$  and  $f_n \circ g = f_n \circ g_\varepsilon$ .

COROLLARY. *For any  $x \in Q$  the compactum  $f_n^{-1}(x)$  satisfies the condition  $DD^nP$ , and also  $X$  satisfies this condition.*

LEMMA 7. *The compactum  $X$  is an absolute extensor in dimension  $n$ .*

PROOF. By a theorem of Kuratowski [9], it suffices to prove that  $X \in LC^{n-1} \cap C^{n-1}$ . The softness of  $f_n$  implies that  $X \in LC^{n-2} \cap C^{n-2}$ . The connectedness of  $X$  in dimension  $n-1$  follows from the polyhedral  $n$ -softness of the mapping  $f_n$ . Let  $\varepsilon > 0$ . There exist an  $i \in \mathbb{N}$  and a  $\gamma > 0$  such that the inverse image  $\alpha_i^{-1}(F)$  of every set  $F \subset N_i$  of diameter less than  $\gamma$  is less than  $\varepsilon$ . Since  $N_i$  is a  $Q$ -manifold, there exists a  $\delta > 0$  such that every continuous mapping of the  $(n-1)$ -dimensional sphere  $S^{n-1}$  into  $N_i$  with range of diameter less than  $\delta$  has a continuous extension to  $B^n$  with range of diameter less than  $\gamma$ . Suppose now that  $g: S^n \rightarrow X$  is a continuous mapping with range of diameter less than  $\delta$ ; then the mapping  $\alpha_i \circ g$  has range of diameter less than  $\delta$ . Consequently, there exists an extension  $q: B^n \rightarrow N_i$  with range of diameter less than  $\gamma$ . Since  $\alpha_i$  is polyhedrally  $n$ -soft, there exists a lifting  $\bar{q}: B^n \rightarrow X$  which extends  $g$  and has range of diameter less than  $\varepsilon$ . The local connectedness of  $X$  in dimension  $n-1$  is proved.

The next lemma precedes the proofs of Lemmas 1 and 2.

LEMMA 8. *Let  $\eta: Y \rightarrow S^n$  be a locally trivial bundle with base the  $n$ -dimensional sphere, let  $A$  be a closed subset of a space  $Z$  with  $\dim A \leq n-1$ , and let  $\psi: Z \rightarrow S^n$  be a given continuous mapping. Then there exists a neighborhood  $W \supset A$  such that the bundle  $\eta': Y' \rightarrow W$  induced by the mapping  $\psi|_W$  is trivial.*

PROOF. Obviously,  $\psi|_A$  is homotopic to a constant. Consequently, there exists a neighborhood  $W \supset A$  such that  $\psi|_W$  is homotopic to a constant mapping; but this implies (see, for example, [17]) that  $\psi|_W$  induces a trivial bundle.

PROOF OF LEMMA 1. We recall the notation:

$$B^{n+1} = \{x \in E^{n+1}: |x| \leq 1\}, \quad B_0^{n+1} = \{x \in E^{n+1}: |x| \leq \frac{1}{2}\},$$

$$S^n = \partial B^{n+1} \quad \text{and} \quad R_n(x) = \{y \in S^n: (x, y) \geq 4|x|^2 - 3|x|\}.$$

For  $x \in B^{n+1} \setminus B_0^{n+1}$  let  $C(x)$  denote the boundary of  $R_n(x)$  on  $S^n$ . For  $x \in \partial B_0^{n+1}$  let  $C(x) = \{-x/|x|\}$ . Denote by  $\rho(x, y)$  the angular distance between  $x$  and  $y$  on the sphere  $S^n$ .

Suppose that  $\dim Z = n$ ,  $A$  is a closed subset of  $Z$  with  $\dim A = n - 2$ ,  $\psi: Z \rightarrow B^{n+1}$  and  $\varphi: A \rightarrow \Gamma_{R_n}$  are continuous mappings, and  $p_n \circ \varphi = \psi|_A$ . Define  $\xi = q_n \circ \varphi$ . We construct a multivalued mapping  $\Phi: Z \rightarrow S^n$  such that  $\Phi(z) \subset R_n(\psi(z))$  for any  $z \in Z$  and  $\xi(z) \in \Phi(z)$  for  $z \in A$ .

Let  $Z_1 = Z \setminus \psi^{-1}(\text{Int } B_0^{n+1})$ ,  $A_1 = A \cap Z_1$ , and

$$G = \{z \in A_1: \xi(z) = -\psi(z)/|\psi(z)|\}.$$

There exists a neighborhood  $V \supset G$  such that for any  $z \in V$

$$\rho(-\psi(z)/|\psi(z)|, C(\psi(z))) < \pi/4$$

and for any  $z \in V \cap A_1$

$$\rho(-\psi(z)/|\psi(z)|, \xi(z)) < \pi/4.$$

Let  $\eta: M_1 S^n \rightarrow S^n$  be the bundle of unit vectors of the tangent bundle on  $S^n$ . By Lemma 8, there exists a neighborhood  $W \supset G$  ( $W \subset V$ ) such that the induced bundle  $\eta': X' \rightarrow W$  is trivial. Suppose that  $Z_2 = W \setminus G$  and  $A_2 = (W \cap A_1) \setminus G$ . Note that the set  $A_2$  is closed in  $Z_2$ . Let  $\pi(x)$  denote the  $n$ -dimensional hyperplane in  $E^{n+1}$  tangent to  $S^n$  at a point  $x \in S^n$ , and let  $\text{pr}_\pi$  be the orthogonal projection onto  $\pi$ . We define a mapping  $\theta: A_2 \rightarrow M_1 S^n$  by setting

$$\theta(z) = \left( \frac{\text{pr}_{\pi(-\psi(z)/|\psi(z)|)}(\xi(z) + \psi(z)/|\psi(z)|)}{\left| \text{pr}_{\pi(-\psi(z)/|\psi(z)|)}(\xi(z) + \psi(z)/|\psi(z)|) \right|}, -\frac{\psi(z)}{|\psi(z)|} \right)$$

for any  $z \in A_2$ . The mapping  $\theta$  determines a mapping  $\nu: A_2 \rightarrow X' \simeq W \times S^{n-1}$  with  $\eta' \circ \nu = \text{id}_{A_2}$ ; that is,  $\nu$  is a section. Let  $\alpha: A_2 \rightarrow S^{n-1}$  be the composition of  $\nu$  and the projection on  $S^{n-1}$ . By Theorem 3.2 in [7], we have that  $\text{AE}(n - 1) = \text{AE}(n - 2, n)$ , and since  $S^{n-1} \in \text{AE}(n - 1)$ , the mapping  $\alpha$  has an extension  $\beta: Z_2 \rightarrow S^{n-1}$ ; consequently,  $\theta$  has an extension  $\mu: Z_2 \rightarrow M_1 S^n$  such that  $\eta \circ \mu = \psi|_{Z_2}$ .

Let  $f: W \rightarrow \mathbf{R}$  be a nonnegative function such that  $f^{-1}(0) = G$ ,  $f(z) = \rho(-\psi(z)/|\psi(z)|, \xi(z))$  for any  $z \in A_2$ , and  $f(z) < \pi/2$  for all  $z \in W$ . Let

$$g(z) = \max\{f(z), \rho(-\psi(z)/|\psi(z)|, C(\psi(z)))\} < \pi/2.$$

For any  $z \in Z_2$  we define  $\gamma(z) \in S^n$  as the point obtained from  $-\psi(z)/|\psi(z)|$  by rotating through an angle  $g(z)$  in the direction of the tangent vector  $\mu(z)$  at the point  $-\psi(z)/|\psi(z)|$ , and we set  $\gamma(z) = -\psi(z)/|\psi(z)|$  for  $z \in G$ . It is easy to see that  $\gamma: W \rightarrow S^n$  is continuous and that  $\gamma(z) \in R_n(\psi(z))$  for any  $z \in W$ , with  $\gamma|_{W \cap A_1} = \xi|_{W \cap A_1}$ . It can be assumed without loss of generality that  $Z$  is a compactum, and the neighborhood  $W$  can then be assumed to be compact. Let

$$a = \min_{z \in A_1 \setminus \text{Int } W} \rho(-\psi(z)/|\psi(z)|, \xi(z)) > 0.$$

We define  $\Phi(z)$  by

$$\Phi(z) = \begin{cases} R_n(\psi(z)) \cap \{x: \rho(x, \psi(z)/|\psi(z)|) \leq \pi - \alpha\} & \text{if } z \in Z_1 \setminus W, \\ \gamma(z) & \text{if } z \in W, \\ S^n & \text{if } z \in Z \setminus Z_1. \end{cases}$$

The multivalued mapping  $\Phi$  is lower semicontinuous, and the family  $\{\Phi(z)\}_{z \in Z}$  is connected and locally equi-connected in dimension less than or equal to  $n - 1$ . It is easy to see that  $\xi(z) \in \Phi(z)$  for any  $z \in A$ . By a theorem of Michael [18], there exists a selection  $\varepsilon: Z \rightarrow S^n$  such that  $\varepsilon|_A = \xi$ . The diagonal product  $\varepsilon\Delta\psi$  is the desired lifting of  $\psi$ .

Before proving Lemma 2 we need two more lemmas.

LEMMA 9. For any continuous mapping  $f: Y \rightarrow B^{n+1}$  of a closed subset of the  $n$ -dimensional sphere  $S^n$  into the  $(n + 1)$ -dimensional ball there exists a homotopy  $H_t: Y \times I \rightarrow B^{n+1}$  that is fixed on  $f^{-1}(\partial B^{n+1})$  and carries  $f$  into a mapping with range in the boundary  $\partial B^{n+1}$  of the ball in such a way that  $H_t(Y)$  does not touch the center of the ball for any  $t > 0$ .

PROOF. Let  $B^{n+1} = \{x \in E^{n+1}: |x| \leq 1\}$ . The case when  $0 \notin f(Y)$  is trivial; we consider the case when  $0 \in f(Y)$ . Let  $B_0^{n+1} = \{x \in E^{n+1}: |x| \leq 1/2\}$  and let  $U = f^{-1}(B_0^{n+1} \setminus \{0\})$ . The set  $U$  is a proper subset of  $S^n$ ; consequently, its  $n$ -dimensional cohomology is equal to zero. We now show that the mapping  $f|_U: U \rightarrow B_0^{n+1} \setminus \{0\}$  is homotopic to a constant. Obviously, the set  $B_0^{n+1} \setminus \{0\}$  is homeomorphic to the product  $S^n \times (0, 1/2]$ ; fix a canonical homeomorphism carrying  $S^n \times \{t\}$  onto  $\{x \in E^{n+1}: |x| = t\}$  for any  $t \in (0, 1/2]$ . Assume the contrary; that is, assume that  $f|_U$  is not homotopic to a constant mapping. Since the projection  $\pi: S^n \times (0, 1/2] \rightarrow S^n$  is a homotopy equivalence, the composition  $\pi \circ f|_U$  is also not homotopic to a constant mapping. But then the mapping  $\pi \circ f|_U: U \rightarrow S^n \subset K(\mathbb{Z}, n)$  gives a nonzero  $n$ -dimensional cohomology class, a contradiction.

Let  $\xi: S^n \times (0, 1/2] \rightarrow (0, 1/2]$  be the projection. We imbed the half-open interval  $(0, 1/2]$  isometrically in  $B_0^{n+1}$  as the subset  $\{x \in E^{n+1}: x = te, t \in (0, 1/2]\}$ , where  $e = (1, 0, 0, \dots, 0) \in E^{n+1}$ . Obviously, the mapping  $\xi$  is homotopic to a constant. Consequently, the mapping  $\xi \circ f|_U$  is homotopic to a constant. Thus,  $f|_U$  and  $\xi \circ f|_U$  are homotopic. Let  $G_t: U \times I \rightarrow B_0^{n+1} \setminus \{0\}$  be a homotopy connecting them, with  $G_0 = f|_U$  and  $G_1 = \xi \circ f|_U$ . Note that for any  $y \in U$  the images  $G_0(y)$  and  $G_1(y)$  lie on the same "parallel" (i.e.,  $\xi G_0(y) = \xi G_1(y)$ ), and thus  $G_t$  can be assumed to satisfy  $\xi \circ G_t(y) = \xi \circ G_0(y)$ . The homotopy  $G_t$  can be extended to a continuous homotopy  $F_t: (U \cup f^{-1}(0)) \times I \rightarrow B_0^{n+1}$  by setting  $F_t(x) = 0$  for any  $t$  and any  $x \in f^{-1}(0)$ . Further,  $F_t$  can be extended to a homotopy  $H'_t: Y \times I \rightarrow B^{n+1}$  such that

$$H'_t(f^{-1}(B^{n+1} \setminus B_0^{n+1})) \subset B^{n+1} \setminus \text{Int } B_0^{n+1}$$

and  $H'_t$  is fixed on  $\partial B^{n+1}$ . The range of the mapping  $H'_t$ , intersected with the ball  $B_0^{n+1}$ , gives a segment that is a radius in  $B_0^{n+1}$ . If after application of  $H'_t$  we contract this segment along itself to the end lying on  $\partial B_0^{n+1}$ , and then use a squeezing of the spherical annulus  $B^{n+1} \setminus \text{Int } B_0^{n+1}$  onto the sphere  $\partial B^{n+1}$ , then we get a certain homotopy  $\bar{H}_t: Y \times I \rightarrow B^{n+1}$ , where  $I = [0, 1]$ . Note that  $\bar{H}_t^{-1}(0) = \bar{H}_0^{-1}(0) \times I_1$ , where  $I_1 = [0, b]$  and  $b < 1$ . A partition of the space  $Y \times I$  into the segments  $\{\{y\} \times I_1\}_{y \in \bar{H}_0^{-1}(0)}$  and points gives a CE-mapping  $\mu: Y \times I \rightarrow Z$ . Obviously,  $Z$  is naturally homeomorphic to the product  $Y \times I$ , and the mapping  $\mu$  divides the mapping  $\bar{H}_t$ ; that is, there exists a mapping  $H_t: Y \times I \rightarrow B^{n+1}$  such that  $\bar{H}_t = H_t \circ \mu$ . The homotopy  $H_t$  is the desired one.

LEMMA 10. *There exists a  $\delta > 0$  such that, for every continuous mapping  $\psi: B^{n+1} \rightarrow B^{n+2}$  with range of diameter less than  $\delta$ , every partial lifting  $\varphi: \partial B^{n+1} \rightarrow \Gamma_{R_{n+1}}$  has an extension  $\bar{\varphi}: B^{n+1} \rightarrow \Gamma_{R_{n+1}}$ .*

PROOF. We take  $1/6$  as  $\delta$ . Suppose that  $\psi$  and  $\varphi$  are given, with  $\text{diam} \psi(B^{n+1}) < 1/6$ . If  $\psi(B^{n+1}) \cap \partial B_0^{n+2} = \emptyset$ , then the lifting  $\varphi$  has an extension. In this case either  $\psi(B^{n+1}) \subset B_0^{n+2}$  or  $\psi(B^{n+1}) \subset B^{n+2} \setminus B_0^{n+2}$ . Over  $B_0^{n+2}$  the mapping  $p_{n+1}$  is a locally trivial bundle with fiber the sphere  $S^{n+1}$ ; that is,  $p_{n+1}$  is an  $(n+1)$ -soft mapping, and  $p_{n+1}$  is absolutely soft over  $B^{n+2} \setminus B_0^{n+2}$ . We consider the case when  $\psi(B^{n+1}) \cap \partial B_0^{n+2} \neq \emptyset$ . Then  $\psi(B^{n+1})$  does not contain the center of  $B^{n+2}$ . Let  $J$  denote the central projection of the set  $B^{n+2} \setminus \{0\}$  onto the sphere  $\partial B^{n+2} = S^{n+1}$ . Then  $J(\psi(B^{n+1}))$  is contained in a ball of angular radius  $\pi/6$  on  $S^{n+1}$ . Let  $z$  be the center of this ball. A simple computation shows that for any  $x \in \psi(B^{n+1})$  the image  $R_{n+1}(x)$  is a ball on  $S^{n+1}$  about  $x/|x|$  of radius at least  $\pi/2$ . Let  $w: S^{n+1} \times B^{n+2} \rightarrow B^{n+2}$  be the projection. By definition,  $\Gamma_{R_{n+1}} \subset S^{n+1} \times B^{n+2}$ ; let  $\xi: S^{n+1} \times B^{n+1} \rightarrow B^{n+1}$  be the bundle induced by the mapping  $\psi$ . We define the section  $\eta: B^{n+1} \rightarrow S^{n+1} \times B^{n+1}$  as follows: for any  $x \in B^{n+1}$  let  $\eta(x) = (-\psi(x)/|\psi(x)|, x)$ . Since the angle between the point  $-\psi(x)/|\psi(x)|$  and  $-z$  is less than  $\pi/6$  for any  $x \in B^{n+1}$ , there exists a fiberwise isometry  $h: S^{n+1} \times B^{n+1} \rightarrow S^{n+1} \times B^{n+1}$  carrying the section  $\eta$  into the trivial section  $\theta: B^{n+1} \rightarrow S^{n+1} \times B^{n+1}$  determined by the point  $-z: \theta(x) = (-z, x)$ . Let  $D^{n+1}$  be the ball of radius  $\pi/2$  about  $z$  on  $S^{n+1}$ . The mapping  $\varphi: \partial B^{n+1} \rightarrow \Gamma_{R_{n+1}}$  induces a section  $\alpha: \partial B^{n+1} \rightarrow S^{n+1} \times \partial B^{n+1}$ , with  $\alpha(x) \in R_{n+1}(\psi(x))$  for any  $x \in \partial B^{n+1} = S^n$ . By Lemma 9, the mapping  $h \circ \alpha$  can be carried into a mapping  $f: S^n \rightarrow S^{n+1} \setminus \text{Int } D^{n+1}$  by a homotopy  $H_t$  such that  $H_t(x) \neq -z$  for all  $t > 0$ . Let  $G_t: S^n \times [0, 1] \rightarrow S^{n+1}$  be a homotopy carrying  $h \circ \alpha$  into a constant mapping ( $G_1(S^n) = d$ ) and obtained from  $H_t$  by adding a contraction of the set  $S^{n+1} \setminus \text{Int } D^{n+1}$  with respect to itself. For any  $x \in B^{n+1}$  let  $r_x$  be a mapping that on  $R_{n+1}(\psi(x))$  is the identity and in  $S^{n+1} \setminus R_{n+1}(\psi(x))$  is a sweeping out, with center at  $-\psi(x)/|\psi(x)|$ , onto the sphere

$$\Sigma_x = \{ y \in S^{n+1}: (\psi(x), y) = 4|\psi(x)|^2 - 3|\psi(x)| \}$$

if the sphere  $\Sigma_x$  is defined; otherwise let  $r_x$  denote the identity mapping on  $S^{n+1}$ . Note that if  $\psi(x) \in \partial B_0^{n+2}$ , then  $|\Sigma_x| = 1$ , while if  $\psi(x) \in \text{Int } B_0^{n+2}$ , then  $\Sigma_x = \emptyset$ .

Let  $B^{n+1} = \{x \in E^{n+1}: |x| \leq 1\}$ . We define an extension  $\bar{\alpha}: B^{n+1} \rightarrow S^{n+1} \times \partial B^{n+1}$  by setting

$$\bar{\alpha}(x) = \begin{cases} r_x \circ h^{-1}(G_{1-|x|}(x/|x|), x) & \text{if } x \neq 0, \\ r_x \circ h^{-1}(d, x) & \text{if } x = 0. \end{cases}$$

Note that  $\bar{\alpha}$  is well defined, and  $\bar{\alpha}(x) \in R_{n+1}(\psi(x))$ . Let  $\psi'$  be the projection parallel to  $\psi$  in the diagram

$$\begin{array}{ccc} S^{n+1} \times B^{n+1} & \xrightarrow{\psi'} & S^{n+1} \times B^{n+2} \\ \downarrow \xi & & \downarrow w \\ B^{n+1} & \xrightarrow{\psi} & B^{n+2} \end{array}$$

Then we define  $\bar{\varphi} = \psi' \circ \bar{\alpha}$ . The lemma is proved.

PROOF OF LEMMA 2. It suffices to prove that if we have a continuous mapping  $\psi: B^{n+1} \rightarrow B^{n+2}$  of the  $(n + 1)$ -dimensional ball and a partial lifting  $\varphi: \partial B^{n+1} \rightarrow \Gamma_{R_{n+1}}(p_{n+1} \circ \varphi = \psi|_{\partial B^{n+1}})$  of it, then there exists an extension  $\bar{\varphi}: B^{n+1} \rightarrow \Gamma_{R_{n+1}}$ , and  $p_{n+1} \circ \bar{\varphi} = \psi$ . Assuming the mappings  $\psi$  and  $\varphi$  to be given, we triangulate  $B^{n+1}$  in such a way that the diameter of the image of any simplex is less than  $1/6$  ( $B^{n+2} = \{x \in E^{n+2}; |x| \leq 1\}$ ). Let  $B_{(n)}^{n+1}$  denote the  $n$ -dimensional skeleton of this triangulation. Since  $p_{n+1}$  is  $n$ -soft (see [7]), there exists an extension  $\varphi': B_{(n)}^{n+1} \rightarrow \Gamma_{R_{n+1}}$ . Application of Lemma 10 concludes the proof.

LEMMA 11. *The following conditions are equivalent for a continuous mapping  $f: X \rightarrow Y$  of compacta when  $n > 0$ :*

- 1)  $f$  is  $n$ -soft.
- 2)  $f$  is  $(n - 1, n)$ -soft.

PROOF. Obviously, 1) implies 2). Suppose that  $f: X \rightarrow Y$  is  $(n - 1, n)$ -soft. This implies that  $f$  is  $(n - 1)$ -soft, and all the fibers of  $f$  are  $AE(n)$ -compacta. To conclude the proof it suffices, by Lemma 1.1 in [7], to verify that the family of all fibers of  $f$  is locally equi-connected in dimension  $n - 1$ . Assume not; that is, assume that there exists a sequence of mappings  $\varphi_k: S^{n-1} \rightarrow f^{-1}(y_k)$  with ranges of diameter tending to zero as  $k$  goes to  $\infty$  and such that each  $\varphi_k$  cannot be extended to the ball with range of diameter less than some positive number  $a$ . Since  $X$  and  $Y$  are compact, it can be assumed that the sequence  $\{y_k\}$  converges to  $y$ , while the sequence of sets  $\varphi_k(S^{n-1})$  converges to some point  $x$ . Obviously,  $f(x) = y$ . Let  $Z$  denote a sequence of balls converging to a point, compactified by this point:  $Z = \bigcup_{k=1}^{\infty} B_k^n \cup \{z\}$ . Let  $A = \bigcup_1^{\infty} \partial B_k \cup \{z\}$ . The sequence  $\{\varphi_k\}$  of mappings determines a continuous mapping  $\varphi: A \rightarrow X$ , with  $\varphi(z) = x$  and  $\varphi|_{\partial B_k^n} = \varphi_k$ . We define a mapping  $\psi: Z \rightarrow Y$  as follows. Let  $\psi(z) = y$  and  $\psi(B_k^n) = y_k$  for any  $k$ . Since  $f$  is  $(n - 1, n)$ -soft, the partial lifting  $\varphi$  has an extension for the mapping  $\psi$ . Since the lifting is continuous at  $z$ , some of the mappings  $\varphi_k$  can be extended to the whole ball with small range, a contradiction.

PROOF OF THEOREM 1. By Lemmas 5–7 and Bestvina’s theorem, the compactum  $X$  is homeomorphic to the Menger universal compactum  $M_n$ . For any  $x \in Q$  the complete inverse image  $f_n^{-1}(x)$  is  $n$ -dimensional, and satisfies  $DD^nP$  by the corollary to Lemma 6. The fact that the fiber  $f_n^{-1}(x)$  is homeomorphic to a Menger compactum is obtained as a corollary to Theorem 2. Conditions 2) and 4) follow from Lemma 4, and condition 3) is a consequence of 2) and Lemma 11. Part 5) is proved by Lemma 6, and it remains to prove 6). Let  $g: Y \rightarrow Z$  be a continuous mapping of an  $n$ -dimensional separable metric space  $Y$ , and let  $i$  be an arbitrary imbedding of  $Z$  in a Hilbert cube  $Q$ . It can be assumed that  $Y$  lies in a Hilbert cube  $Q_1$  and that  $g$  is the restriction of the projection  $\pi: Q_1 \rightarrow Q$  to a factor (it suffices to consider the graph of  $g$ ). Since  $\alpha_1$  is  $(n - 2, n)$ -soft, there exists a mapping  $j: Y \rightarrow M_n$  such that  $\alpha_1 \circ j = \text{id}_Y$ . We remark that  $j$  is univalent, and  $j^{-1}$  is continuous as a restriction of the continuous mapping  $\alpha_1$ . Thus, there exist imbeddings  $j: Y \hookrightarrow M_n$  and  $i: Z \hookrightarrow Q$  such that  $i \circ g = f_n \circ j$ .

THEOREM 2. *For any  $LC^{n-1}$ -compactum  $X \subset Q$  the complete inverse image  $f_n^{-1}(X)$  is an  $n$ -dimensional Menger manifold. But if  $X \in AE(n)$ , then the inverse image  $f_n^{-1}(X)$  is homeomorphic to the Menger compactum  $M_n$ .*

The proof is based on a number of auxiliary assertions.

Let  $p_n: \Gamma_{R_n} \rightarrow B^{n+1}$  be the projection of the graph of the retraction described above, and let  $B^{n+1} = \{x \in E^{n+1}; |x| \leq 1\}$ .

LEMMA 12. For any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for any continuous mapping  $\varphi: S^{n-1} \rightarrow \Gamma_{R_n}$  of the  $(n - 1)$ -dimensional sphere with range of diameter less than  $\delta$  and for any  $\delta$ -homotopy  $H_i: S^{n-1} \times I \rightarrow B^{n+1}$ ,  $H_0 = p_n \circ \varphi$ , there exists a lifting  $G_i: S^{n-1} \times I \rightarrow \Gamma_{R_n}$  such that  $G_0 = \varphi$  and  $\text{diam } G_i(S^{n-1} \times I) < \varepsilon$ .

The proof is actually contained in the proof of Lemma 10 and is thus omitted.

LEMMA 13. Suppose that  $Y$  is an  $LC^{n-1}$ -compactum,  $p^{n+1}: \Gamma_{n+1} \rightarrow K^{(n+1)}$  is the projection in Lemma 3, and  $\varphi: Y \rightarrow K^{(n+1)}$  is a continuous mapping. Then the fiberwise product  $X$  of the spaces  $\Gamma_{n+1}$  and  $Y$  with respect to the mappings  $p^{n+1}$  and  $\varphi$  is an  $LC^{n-1}$ -compactum.

PROOF. Let  $\alpha(\bigcup_{i=1}^\infty B_i^n)$  be the Aleksandrov compactification of a discrete family of  $n$ -dimensional balls, and let  $x$  be the compactifying point. Denote by  $S_i^{n-1}$  the boundary of  $B_i^n$ .

Assume the opposite: suppose that there exists a continuous mapping  $\psi: \alpha(\bigcup_{i=1}^\infty S_i^{n-1}) \rightarrow X$  such that for every  $i$  every extension of  $\psi|_{S_i^{n-1}}$  to  $B_i^n$  has range of diameter greater than a fixed  $a > 0$ . Let  $\sigma$  be an  $(n + 1)$ -dimensional simplex in  $K^{(n+1)}$  containing the point  $y = p^{n+1}\varphi\psi(x)$ . The simplex  $\sigma$  will be identified with the  $(n + 1)$ -dimensional unit ball  $B^{n+1}$ , and the restriction of the multivalued retraction  $R^{n+1}: K^{(n+1)} \rightarrow K^{(n)}$  to  $\sigma$  will be identified with the multivalued retraction  $R_n: B^{n+1} \rightarrow \partial B^{n+1}$  described at the beginning.

1) Let  $y \notin \partial B_0^{n+1}$ . Then  $K^{(n+1)}$  contains a neighborhood  $V \ni y$  not touching  $\partial B_0^{n+1}$ . The restriction of  $p^{n+1}$  to this neighborhood is  $n$ -soft, in view of Lemma 1.1 of [7]. But then the restriction of  $q$  to  $q^{-1}(\varphi^{-1}(V))$  is also  $n$ -soft, and this and the local connectivity of  $Y$  yield a contradiction.

2) Suppose that  $y \in \partial B_0^{n+1}$ . Since  $Y \in LC^{n-1}$ , there exists a continuous extension  $\xi: \alpha(\bigcup_{i=1}^\infty B_i^n) \rightarrow Y$  of  $q \circ \psi$ . There exists an  $\varepsilon > 0$  such that for every  $F \subset X$  the inequalities  $\text{diam}(\varphi(F)) < \varepsilon$  and  $\text{diam}(qF) < \varepsilon$  imply that  $\text{diam } F < a$ . Suppose that  $\delta$  has been chosen with respect to  $\varepsilon$  as in Lemma 12. Let  $b_i$  be the center of  $B_i^n$ ; we join the points  $\xi(b_i)$  and  $q\psi(x)$  by paths  $\eta_i: I = [0, 1] \rightarrow Y$  in such a way that  $\lim_{i \rightarrow \infty} \text{diam } \eta_i(I) = 0$ . There is a  $k$  such that

$$\max\{\text{diam}(\eta_k(I) \cup \xi(B_k^n)), \text{diam}(\varphi\eta_k(I) \cup \varphi\xi(B_k^n))\} < \delta.$$

Let  $H_i: S_k^{n-1} \times I \rightarrow Y$  be a homotopy joining  $\xi|_{S_k^{n-1}}$  with the constant mapping into  $\xi(b_k)$  with respect to the image  $\xi(B_k^n)$ , and then carrying this constant mapping along the path  $\eta_k$  into another constant mapping—into the point  $q\psi(x)$ . By Lemma 12, the homotopy  $\varphi \circ H_i$  has a covering

$$G_i: S_k^{n-1} \times I \rightarrow \Gamma_{n+1}$$

with  $G_0 = \varphi' \circ \psi$ ,  $G_1(S_k^{n-1}) \subset (p^{n+1})^{-1}(y)$ , and  $\text{diam}(G_i(S_k^{n-1} \times I)) < \varepsilon$ . Since  $(p^{n+1})^{-1}(y) \simeq S^n$  and the convex hull of  $G_1(S_k^{n-1})$  on the sphere  $S^n$  has diameter less than  $\varepsilon$ , the homotopy  $G_i$  can be extended to an  $\varepsilon$ -homotopy  $L_i: S_k^{n-1} \times I \rightarrow \Gamma_{n+1}$  into a constant mapping by adding a contraction along itself of the convex hull (on the sphere)  $\text{conv } G_1(S_k^{n-1})$  into a point. After extending the homotopy  $H_i$  by a constant mapping, we

can get a homotopy  $N_i: S_k^{n-1} \times I \rightarrow Y$  such that  $\varphi \circ N_i = p^{n+1} \circ L_i$ . But then an  $\alpha$ -homotopy  $M_i: S_k^{n-1} \times I \rightarrow X$  such that  $M_0 = \psi|_{S_k^{n-1}}$  and  $M_1(S_k^{n-1}) = \psi(x)$  has been defined, a contradiction.

LEMMA 14. *Let  $K$  be a compact polyhedron of dimension  $\dim K = m > n$ , and  $f: L \rightarrow K$  the mapping constructed in Lemma 3. Then for any  $LC^{n-1}$ -compactum  $Y$  and any continuous mapping  $\psi: Y \rightarrow K$  the fiberwise product  $X$  of the spaces  $L$  and  $Y$  with respect to the mappings  $f$  and  $\psi$  is an  $LC^{n-1}$ -compactum.*

PROOF. Let  $L_1$  be the inverse limit of the diagram

$$\begin{array}{ccccccc}
 \Gamma_m & & \Gamma_{m-1} & & \cdots & & \Gamma_{n+2} \\
 \downarrow & \searrow & \downarrow & \searrow & & & \downarrow & \searrow \\
 K^{(m)} & & K^{(m-1)} & & \cdots & & K^{(n+2)} & & K^{(n+1)},
 \end{array}$$

and let  $p_i: L_1 \rightarrow K^{(i)}$  and  $q_i: L_1 \rightarrow \Gamma_i$  be its projections. Since the projections  $p^i$  are  $n$ -soft, we get from the lemma on parallels in [7] for  $n$ -soft mappings that the fiberwise product  $Y'$  of  $L_1$  and  $Y$  with respect to the mappings  $p_m$  and  $\psi$  is an  $LC^{n-1}$ -compactum. Let  $\varphi': Y' \rightarrow L_1$  be the projection parallel to  $\varphi$ . It is easy to see that  $L$  is the fiberwise product of  $L_1$  and  $\Gamma_{n+1}$  with respect to the mappings  $p_{n+1}: L_1 \rightarrow K^{(n+1)}$  and  $p^{n+1}: \Gamma_{n+1} \rightarrow K^{(n+1)}$ . Then  $X$  is the fiberwise product of the spaces  $\Gamma_{n+1}$  and  $Y'$  with respect to the mappings  $p^{n+1}: \Gamma_{n+1} \rightarrow K^{(n+1)}$  and  $p_{n+1} \circ \varphi': Y' \rightarrow K^{(n+1)}$ . By Lemma 13,  $X$  is an  $LC^{n-1}$ -compactum.

LEMMA 15. *For any  $LC^{n-1}$ -compactum  $Y \subset Q$  the complete inverse image  $f_n^{-1}(Y)$  is an  $LC^{n-1}$ -compactum.*

PROOF. For any  $Y \subset Q$  the complete inverse image  $f_n^{-1}(Y)$  is the limit of the inverse spectrum  $\{X_i, \alpha_i^{i+1}\}$ , where  $X_1 = \pi^{-1}(Y)$  and  $X_{i+1} = (\alpha_i^{i+1})^{-1}(X_i)$ . We show by induction that  $X_i \in LC^{n-1}$ . The compactum  $X_1$  is in  $LC^{n-1}$ , being the product of the  $LC^{n-1}$ -compactum  $Y$  and the Hilbert cube. Suppose that  $X_i \in LC^{n-1}$ . The projection  $\alpha_i^{i+1}$  decomposes into the composition  $\gamma_{i+1} \circ \eta_{i+1}$ , where  $\eta_{i+1}$  is the projection parallel to the Hilbert cube, and  $\gamma_{i+1}$  is the projection parallel to  $\tilde{f}_{i+1}: L_{i+1} \rightarrow K_i$  in the fiberwise product of  $L_{i+1}$  and  $X_i$  with respect to the mapping  $\tilde{f}_{i+1}: L_{i+1} \rightarrow K_i$  defined by Lemma 3 and the projection mapping  $\omega^i: Q^i \times K_i \rightarrow K_i$ , restricted to  $X_i \subset Q^i \times K_i = M_i$ . By Lemma 14, the compactum  $\eta_{i+1}(X_{i+1})$  belongs to the class  $LC^{n-1}$ ; but then  $X_{i+1}$  also belongs to  $LC^{n-1}$ .

Suppose now that  $\varepsilon > 0$  is given. There is an  $i$  such that every inverse image  $\alpha_i^{-1}(A)$  of a set of diameter less than  $\kappa > 0$  has diameter less than  $\varepsilon$ . Since  $f_n^{-1}(Y)$  is compact, there exists a  $\delta > 0$  such that every set  $B$  of diameter less than  $\delta$  has image  $\alpha_i(B)$  with diameter less than some  $\sigma$  chosen according to  $\kappa$ . Suppose now that  $\varphi: S^{n-1} \rightarrow f_n^{-1}(Y)$  is a given mapping of the  $(n - 1)$ -dimensional sphere with range of diameter less than  $\delta$ ; then the diameter of the range  $\alpha_i \varphi(S^{n-1})$  is less than  $\sigma$ , and, since  $X_i \in LC^{n-1}$ ,  $\sigma$  can be chosen so that every mapping  $\beta: S^{n-1} \rightarrow X_i$  with range of diameter less than  $\sigma$  has an extension  $\bar{\beta}: B^n \rightarrow X_i$  to the ball with range of diameter less than  $\kappa$ . Let  $\psi: B^{n-1} \rightarrow X_i$  be an extension of  $\alpha_i \circ \varphi$ . Since  $\alpha_i$  is polyhedrally  $n$ -soft, the lifting has an extension  $\bar{\varphi}: B^n \rightarrow f_n^{-1}(Y)$  ( $\bar{\varphi}|_{S^{n-1}} = \varphi$  and  $\alpha_i \circ \bar{\varphi} = \psi$ ), with the diameter of the range  $\bar{\varphi}(B^n)$  less than  $\varepsilon$ .

It is proved that  $f_n^{-1}(Y) \in LC^{n-1}$ .

REMARK. It is possible to prove the stronger assertion that, for any  $LC^{n-1}$ -compactum  $Y$  and any continuous mapping  $\varphi: Y \rightarrow Q$  of it, the fiberwise product of  $Y$  and  $M_n$  with respect to  $\varphi$  and  $f_n$  is an  $LC^{n-1}$ -compactum.

PROOF OF THEOREM 2. Since  $X \in LC^{n-1}$ , the complete inverse image  $f_n^{-1}(X)$  is an  $LC^{n-1}$ -compactum by Lemma 15. It follows from Lemma 6 that  $f_n^{-1}(X)$  satisfies  $DD^n P$ . By Bestvina's theorem,  $f_n^{-1}(X)$  is an  $n$ -dimensional Menger manifold. Suppose now that, in addition,  $X \in AE(n)$ ; then, since  $f_n$  is polyhedrally  $n$ -soft, the inverse image  $f_n^{-1}(X)$  is connected in dimension  $n - 1$  and, consequently,  $f_n^{-1}(X) \in AE(n)$ . By Bestvina's theorem,  $f_n^{-1}(X)$  is homeomorphic to the universal Menger compactum.

COROLLARY. For any  $x \in Q$  the fiber  $f_n^{-1}(x)$  is homeomorphic to the  $n$ -dimensional Menger compactum.

THEOREM 3. For any positive integer  $n$  there exists a continuous mapping  $g_n: M_n \rightarrow M_n$  of the Menger universal compactum onto itself with properties 1)–5) of Theorem 1 and the property

6)  $g_n$  is a universal object in the category of continuous mappings between  $n$ -dimensional compacta.

PROOF. Let  $M_n \subset Q$ . As  $g_n$  we take the restriction of  $f_n$  to the complete inverse image  $f_n^{-1}(M_n)$ , which, by Theorem 2, is homeomorphic to the Menger universal compactum  $M_n$ . The properties 1)–6) of  $g_n$  follow from Theorem 1.

THEOREM 4.1. For any positive integer  $n$  there exists a continuous mapping  $\alpha_n: I^{2n+1} \rightarrow Q$  of the  $(2n + 1)$ -dimensional cube onto the Hilbert cube such that every continuous mapping of an  $n$ -dimensional separable metric space imbeds in  $\alpha_n$ .

PROOF. We imbed the Menger compactum  $M_n$  in  $I^{2n+1}$  and take  $\alpha_n$  to be an arbitrary continuous extension of the mapping  $f_n: M_n \rightarrow Q$  in Theorem 1.

THEOREM 4.2. For any positive integer  $n$  there exists a continuous mapping  $\beta_n: I^{2n+1} \rightarrow I^{2n+1}$  of the  $(2n + 1)$ -dimensional cube onto itself such that every continuous mapping between  $n$ -dimensional separable metric spaces imbeds in  $\beta_n$ .

PROOF. Let  $\gamma_n: X \rightarrow I^{2n+1}$  be the mapping  $f_n$ , restricted to the inverse image  $f_n^{-1}(I^{2n+1})$ , where  $I^{2n+1}$  is assumed to be imbedded in  $Q$ , and  $\beta_n$  is an arbitrary continuous extension of  $\gamma_n$  from  $X$  onto  $I^{2n+1}$ . The universality of  $\beta_n$  follows from Theorem 1.

THEOREM 5.1. The following conditions are equivalent for a metric compactum  $X$ :

- 1)  $X \in AE(n)$ .
- 2)  $X$  is an  $(n - 1)$ -invertible image of the Menger compactum  $M_n$ .
- 3)  $X$  is an  $n$ -invertible image of the Menger compactum  $M_n$ .
- 4)  $X$  is the image of  $M_n$  under an  $(n - 2, n)$ -soft polyhedrally  $n$ -soft mapping whose fibers all are homeomorphic to  $M_n$ .
- 5)  $X$  is an  $(n - 1)$ -invertible image of the Hilbert cube.

PROOF. Theorem 2 yields the implication 1)  $\Rightarrow$  4). The implications 4)  $\Rightarrow$  3)  $\Rightarrow$  2) are trivial. Obviously, an  $(n - 1)$ -invertible image of an  $AE(n)$ -space belongs to the class  $AE(n - 1, n)$ , which (see, for example, Lemma 11) coincides with  $AE(n)$ . Thus, the implications 5)  $\Rightarrow$  1) and 2)  $\Rightarrow$  1) have been obtained. The implication 1)  $\Rightarrow$  5) follows

from the equality  $AE(n) = AE(n - 1, \infty)$  and the fact that every compactum is an  $m$ -invertible image of an  $m$ -dimensional one [7].

The proof of the following result is similar.

**THEOREM 5.2.** *The following conditions are equivalent for a metric compactum  $X$ :*

- 1)  $X \in LC^{n-1}$ .
- 2)  $X$  is an  $(n - 1)$ -invertible image of an  $n$ -dimensional Menger manifold.
- 3)  $X$  is an  $n$ -invertible image of an  $n$ -dimensional Menger manifold.
- 4)  $X$  is the image of an  $M_n$ -manifold under an  $(n - 2, n)$ -soft polyhedrally  $n$ -soft mapping whose fibers are all homeomorphic to  $M_n$ .
- 5)  $X$  is an  $(n - 1)$ -invertible image of a  $Q$ -manifold.

**THEOREM 6** (on stabilization). *For any  $M_n$ -manifold  $X$  and any imbedding of it in the Menger compactum  $M_n$  the space  $g_n^{-1}(X)$  is homeomorphic to  $X$ .*

**PROOF.** The mapping  $g_n|_{g_n^{-1}(X)}: g_n^{-1}(X) \rightarrow X$ , being a  $UV^{n-1}$ -mapping, induces an isomorphism of  $k$ -dimensional homotopy groups for  $k < n$ . By Theorem 2, the space  $g_n^{-1}(X)$  is an  $M_n$ -manifold. A theorem of Bestvina (Theorem 2.8.6 in [6]) gives us that the  $M_n$ -manifolds  $g_n^{-1}(X)$  and  $X$  are homeomorphic.

**THEOREM 7** (on triangulation). *For any  $M_n$ -manifold  $X$  there is an  $n$ -dimensional polyhedron  $K$  such that, for any imbedding of  $K$  in the Menger compactum  $M_n$ , the space  $g_n^{-1}(K)$  is homeomorphic to  $X$ .*

**PROOF.** By a theorem of Bestvina (Proposition 5.1.3 in [6]), there exist a polyhedron  $K$  of dimension  $\dim K \leq n$  and a mapping  $f: K \rightarrow X$  inducing an isomorphism of homotopy groups of dimension less than or equal to  $n - 1$ . Suppose that the polyhedron  $K$  is imbedded in an arbitrary way in  $M_n$ . The composition  $f \circ g_n$  also induces an isomorphism of homotopy groups of dimension less than or equal to  $n - 1$ . By Theorem 2,  $g_n^{-1}(K)$  is an  $M_n$ -manifold, which is homeomorphic to  $X$  by Bestvina's theorem.

The definition of a partial product can be found in [19]. Let  $D$  be a two-point set, and let  $\{O_i\}$  be a family of open subsets of a compactum  $X$ . Let  $P(X, \{O_i\})$  denote the partial product with base  $X$  and with constant fiber  $D$ . By definition,  $P(X, \{O_i\})$  is the fiberwise product of compacta  $X_i$  that are inverse images of at most two-fold mappings  $\alpha_i: X_i \rightarrow X$  with respect to these mappings, while the set of points where  $\alpha_i$  is not one-fold coincides with  $O_i$ , and the restriction  $\alpha_i|_{\alpha_i^{-1}O_i}$  is homeomorphic to the trivial bundle with fiber  $D$ .

**THEOREM 8.** *Every Menger  $n$ -dimensional manifold  $X$  can be represented as a partial product  $P(L, \{O_i\}_1^\infty)$  over an  $n$ -dimensional polyhedron  $L$ .*

**PROOF.** Let  $f: L \rightarrow X$  be a mapping of an  $n$ -dimensional polyhedron that induces an isomorphism of homotopy groups in dimensions less than or equal to  $n - 1$ . Without loss of generality it can be assumed that  $L$  is the  $n$ -dimensional skeleton of some triangulation of a manifold with boundary. The family  $\{O_i\}_1^\infty$  can be chosen as follows: take all the stars in the first barycentric subdivision on  $L$  with respect to the vertices of the original triangulation, then all the stars in the second barycentric subdivision with respect to the vertices of the first, and so on. It can be shown by induction that in this case every projection  $\beta_m: P(L, \{O_i\}_1^m) \rightarrow L$  induces an isomorphism of homotopy groups in dimensions less than or equal to  $n - 1$ . Consequently, the limit projection  $\beta: P(L, \{O_i\}_1^\infty) \rightarrow L$

also induces an isomorphism in dimensions less than or equal to  $n - 1$ . By Bestvina's criterion, the partial product  $P(L, \{O_i\}_1^\infty)$  is an  $M_n$ -manifold, and, by Theorem 2.8.6 in [6], the  $M_n$ -manifold  $X$  is homeomorphic to  $P(L, \{O_i\}_i^\infty)$ .

**COROLLARY.** *The group  $(\mathbf{Z}_2)^{w_0}$  acts effectively on every  $M_n$ -manifold  $X$ , with orbit space homeomorphic to an  $n$ -dimensional polyhedron.*

**PROOF.** By a theorem of Pasynkov (Theorem 12.2 in [19]), the group  $D^{w_0}$  acts on every countable partial product  $P(L, \{O_i\}_1^\infty)$ , with orbit space homeomorphic to the base  $L$ .

**§2. Universal objects and absolute extensors  
in the nonmetrizable case**

**THEOREM 9.** *For any positive integer  $n$  and any cardinal number  $\tau$  there exist a compact Hausdorff space  $D_n^\tau$  and a continuous mapping  $f_n^\tau: D_n^\tau \rightarrow I^\tau$  onto the Tychonoff cube  $I^\tau$ , with the following conditions satisfied:*

- 1) *The dimension of  $D_n^\tau$  is equal to  $n$  ( $\dim D_n^\tau = n$ ), and the weight is equal to  $\tau$ .*
- 2)  *$D_n^\tau \in \text{AE}(n - 1) \cap \text{AE}(n - 2, n) \cap C^{n-1} \cap \text{LC}^{n-1}$ .*
- 3)  *$D_n^\tau$  is universal in the class of spaces of weight  $\tau$  that are  $n$ -dimensional in the Lebesgue sense.*
- 4) *The mapping  $f_n^\tau$  is  $(n - 1)$ -soft,  $(n - 2, n)$ -soft, and polyhedrally  $n$ -soft.*
- 5)  *$f_n^\tau$  is universal in the class of continuous mappings of spaces of weight  $\tau$  that are  $n$ -dimensional in the Lebesgue sense.*

**PROOF.** Let  $\pi: I^\tau \times I^\tau \rightarrow I^\tau$  be the projection, and construct a mapping  $q: D_n^\tau \rightarrow I^\tau$  satisfying conditions 1), 2), and 4). Then the mapping  $\pi \circ q$  satisfies 5) because  $q$  is  $(n - 2, n)$ -soft. It is easy to see that 5) implies 3).

We construct  $D_n^\tau$  and  $f_n^\tau$  by induction on the weight  $\tau$ . Theorem 1 serves as the induction base. The compact Hausdorff space  $D_n^\tau$  is constructed as the limit of a transfinite continuous inverse spectrum  $\{Z_\alpha, p_\alpha^{\alpha+1}, \alpha < \omega(\tau)\}$  that begins with the Menger compactum  $M_n = Z_1$ , and the mapping  $f_n^\tau$  is constructed as the limit of the mappings of the spectra:

$$\{f_\alpha\}: \{Z_\alpha, p_\alpha^{\alpha+1}, \alpha < \omega(\tau)\} \rightarrow \{I^\alpha, \pi_\alpha^{\alpha+1}, \alpha < \omega(\tau)\},$$

where  $I$  denotes the Hilbert cube, and  $\pi_\alpha^{\alpha+1}$  the projection. Both spectra together with the morphism will be constructed in such a way that the following commutative diagram is defined for any  $\alpha < \omega(\tau)$ :

$$\begin{array}{ccccc} Z_\alpha & \xleftarrow{h_\alpha} & \bar{Z}_\alpha & \xleftarrow{q_\alpha} & Z_{\alpha+1} \\ \downarrow f_\alpha & & \downarrow g_\alpha & & \\ I^\alpha & \xleftarrow{\pi_\alpha^{\alpha+1}} & I^{\alpha+1} & = & I^\alpha \times I, \end{array}$$

where  $\bar{Z}_\alpha$  is the fiberwise product of the cube  $I^{\alpha+1}$  and  $Z_\alpha$  with respect to the mappings  $\pi_\alpha^{\alpha+1}$  and  $f_\alpha$ , and  $h_\alpha$  and  $g_\alpha$  are its projections. Moreover, the mappings  $f_\alpha$  and  $q_\alpha$  satisfy all the necessary conditions for softness, and  $g_\alpha \circ q_\alpha = f_{\alpha+1}$  and  $h_\alpha \circ q_\alpha = p_\alpha^{\alpha+1}$ , while  $\dim Z_{\alpha+1} = n$  and  $w(Z_{\alpha+1}) = |\alpha|$ . It is easy to see that the induction hypothesis and Theorem 1 enable us to construct spectra with such a morphism between them. Obviously, the limit space  $D_n^\tau$  satisfies condition 1). It is easy to see that the limit mapping  $f_n^\tau$  has

all the softness properties required in 4), and this implies that  $D_n^\tau \in \text{AE}(n-1) \cap \text{AE}(n-2, n) \cap C^{n-1}$ . It remains to verify the local connectedness of  $D_n^\tau$  in dimension  $n-1$ , which follows from the local connectedness in dimension  $n-1$  of the spaces  $Z_\alpha$  and the polyhedral  $n$ -softness of the projections  $p_\alpha^{\alpha+1}$ .

**COROLLARY.** *There exists a mapping  $g_n^\tau: D_n^\tau \rightarrow D_n^\tau$ , that is universal in the class of mappings between  $n$ -dimensional (in the sense of  $\dim$ ) spaces of weight at most  $\tau$  and satisfies condition 4).*

**THEOREM 10.** *The following conditions are equivalent for any positive integers  $n$  and  $m > n + 1$ :*

- 1)  $X \in \text{AE}(n, m)$ .
- 2)  $X$  is an  $n$ -invertible image of the universal compact Hausdorff space  $D_m^{w(X)}$ .

**PROOF.** The implication 2)  $\Rightarrow$  1) follows from the fact that  $D_m^{w(X)} \in \text{AE}(m-2, m) \subset \text{AE}(n, m)$  for  $n < m - 1$ . Let  $X \in \text{AE}(n, m)$ . As for every compact Hausdorff space,  $X$  is an  $n$ -invertible image of an  $n$ -dimensional compactum,  $\varphi: Z \rightarrow X$  (it suffices to imbed  $X$  in  $I^\tau$  and consider  $Z = (f_n^\tau)^{-1}(X)$ ). The compact Hausdorff space  $Z$  can be assumed to be imbedded in  $D_m^{w(X)}$ , because the latter is universal. Since  $X \in \text{AE}(n, m)$  and  $\dim D_m^{w(X)} = m$ , there exists an extension  $\psi: D_m^{w(X)} \rightarrow X$ , which is  $n$ -invertible, being an extension of an  $n$ -invertible mapping.

Let  $f \Delta g$  denote the diagonal product of mappings  $f: X \rightarrow X_1$  and  $g: X \rightarrow X_2$ . The definition of the  $\sigma$ -spectrum can be found in [13]. An inverse spectrum  $\{X_\alpha; f_\beta^\alpha; \alpha, \beta \in A\}$  over a directed set  $A$  is called a  $\sigma$ -spectrum if  $w(X_\alpha) \leq \aleph_0$  for any  $\alpha$ , and  $\sup\{\alpha_i\}$  exists in  $A$  and

$$X_{\sup\{\alpha_i\}} = \varprojlim \{X_{\alpha_i}; f_{\alpha_i}^{\alpha_{i+1}}\}$$

for any countable chain  $\alpha_1 < \alpha_2 < \dots < \alpha_i < \dots$ .

**DEFINITION.** A  $\sigma$ -spectrum  $S = \{X_\alpha; p_\beta^\alpha, \alpha, \beta \in A\}$  with limit  $X = \varprojlim S$  is said to be *multiplicatively  $n$ -soft* if for any  $B \subset A$  and  $\alpha \in A$  the projection

$$p_B^{(\alpha) \cup B}: \left( \bigtriangleup_B p_\beta \Delta p_\alpha \right) (X) \rightarrow \bigtriangleup_B p_\beta (X)$$

is an  $n$ -soft mapping.

**THEOREM 11.** *A compact Hausdorff space  $X$  is an absolute extensor in dimension  $n$  if and only if it is homeomorphic to the limit  $X = \varprojlim S$  of a multiplicatively  $n$ -soft spectrum in  $\text{AE}(n)$ -compacta.*

**PROOF.** The fact that an  $\text{AE}(n)$ -compact Hausdorff space decomposes into a multiplicatively  $n$ -soft spectrum was actually proved in [7] (Theorem 4.2). Conversely, if a compact Hausdorff space  $X$  is the limit of a multiplicatively  $n$ -soft spectrum  $S = \{X_\alpha; p_\beta^\alpha, A\}$ , then we can use transfinite induction to construct a transfinite continuous inverse spectrum  $\Sigma = \{Y_\alpha; g_\beta^\alpha, w(X)\}$  of length equal to the weight of  $X$ , beginning with an  $\text{AE}(n)$ -compactum, with  $n$ -soft short projections  $g_\alpha^{\alpha+1}$ , and with the same limit  $X$ . The compact Hausdorff space  $X$  is then obviously an  $\text{AE}(n)$ -space.

DEFINITION [20]. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow p & & \downarrow q \\ X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

is said to be *n-soft* if it can be supplemented to form a diagram

$$\begin{array}{ccccc} X & \xrightarrow{h} & \bar{X} & \xrightarrow{f'_1} & Y \\ & & \downarrow q' & & \downarrow q \\ & & X_1 & \xrightarrow{f_1} & Y_1 \end{array}$$

in which all the mappings are *n-soft*,  $p = q' \circ h$ ,  $f = f'_1 \circ h$ , and  $\bar{X}$  is the fiberwise product of the spaces  $X_1$  and  $Y$  with respect to the mappings  $f_1$  and  $q$ , while  $q'$  and  $f'_1$  are the projections parallel to  $q$  and  $f_1$  in the fiberwise product.

The next theorem is a slight refinement of a theorem of Chigogidze [20] and coincides with it in the case when the weight of the spaces is equal to  $\aleph_1$ .

THEOREM 12. Let  $f: X \rightarrow Y$  be an *n-soft* mapping of  $AE(n)$ -compact Hausdorff spaces. Then  $X$  and  $Y$  are limits of multiplicatively *n-soft* spectra  $S_X = \{X_\alpha; p_\beta^\alpha, A\}$  and  $S_Y = \{Y_\alpha; q_\beta^\alpha, A\}$  such that for every  $B \subset A$  a mapping

$$f_B: X_B = \left( \bigtriangleup_{\alpha \in B} P_\alpha \right) (X) \rightarrow Y_B \left( \bigtriangleup_{\alpha \in B} q_\alpha \right) (Y)$$

is defined such that the following diagram is commutative and *n-soft*:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \bigtriangleup_B p_\alpha \downarrow & & \downarrow \bigtriangleup_B q_\alpha \\ X_B & \xrightarrow{f_B} & Y_B \end{array}$$

DEFINITION. The mapping  $f: X \rightarrow Y$  of compact Hausdorff spaces has a *metrizable kernel* if  $X$  is imbeddable in  $Y \times Q$ , and  $f$  is homeomorphic to the restriction of the projection  $\pi: Y \times Q \rightarrow Y$ , where  $Q$  is the Hilbert cube.

LEMMA 16. Suppose that  $\bar{X} = \{f_1: X_1 \rightarrow X_0\} \times \{f_2: X_2 \rightarrow X_0\}$  is the fiberwise product of compact Hausdorff spaces  $X_1$  and  $X_2$  with respect to the mappings  $f_1$  and  $f_2$ , and let  $p_1: \bar{X} \rightarrow X_2$  and  $p_2: \bar{X} \rightarrow X_1$  be its projections. Assume that  $f_1$  is open and has a metrizable kernel, and that  $p_1$  is *n-soft*. Then  $f_1$  is also *n-soft*.

PROOF. Since  $f_1$  has a metrizable kernel,  $X_1$  is contained in the product  $X_0 \times Q$  of a compact Hausdorff space  $X_0$  and the Hilbert cube, and  $f_1$  is the restriction of the projection  $\pi: X_0 \times Q \rightarrow X_0$  to  $X_1$ . Then it can be assumed that  $\bar{X}$  is contained in the product  $X_2 \times Q$ , the restriction of the projection  $\omega: X_2 \times Q \rightarrow X_2$  to  $\bar{X}$  is homeomorphic to  $p_1$ , and the restriction of  $f_2 \times \text{id}_Q: X_2 \times Q \rightarrow X_0 \times Q$  to  $\bar{X}$  coincides with  $p_2$ . Since  $p_1$  is *n-soft*, the family  $\{p_1^{-1}(x)\}_{x \in X_2}$  of fibers is equi- $LC^{n-1}$ . This family of subsets of  $Q$  contains the set of fibers  $\{f_1^{-1}(x)\}_{x \in X_0}$  as a subfamily. Consequently, the family  $\{f_1^{-1}(x)\}_{x \in X_0}$  is also equi- $LC^{n-1}$ . By Lemma 1.1 in [7],  $f_1$  is *n-soft*.

LEMMA 17. *The following conditions are equivalent for a continuous mapping  $f: X \rightarrow Y$  with a metrizable kernel, where  $X$  and  $Y$  are compact Hausdorff spaces:*

- 1)  $f$  is  $(n, n + 1)$ -soft.
- 2)  $f$  is  $(n + 1)$ -soft.

The proof is analogous to that of Lemma 11.

THEOREM 13. *An  $n$ -soft image of an  $AE(n + 1)$ -compact Hausdorff space is an  $AE(n + 1)$ -space.*

PROOF. Suppose that  $f: X \rightarrow Y$  is  $n$ -soft and  $X \in AE(n + 1)$ , and let  $S_X = \{X_\alpha; p_\beta^\alpha, A\}$  and  $S_Y = \{Y_\alpha; q_\beta^\alpha, A\}$  be the multiplicatively  $n$ -soft spectra in Theorem 12. Without loss of generality it can be assumed that  $S_X$  is a multiplicatively  $(n + 1)$ -soft spectrum. We show that  $S_Y$  is multiplicatively  $(n + 1)$ -soft and conclude the proof by Theorem 11.

Let  $B \subset A$  be an arbitrary subset, and let  $\alpha \in A$  be an arbitrary element.

By Theorem 12, the following diagram is defined:

$$\begin{array}{ccccc}
 X & \xrightarrow{h} & \bar{X} & & \xrightarrow{p'_B} & X_B \\
 & & \downarrow f' & & & \downarrow f_B \\
 Y & \xrightarrow{\psi} & Y_{B \cup \{\alpha\}} & \xrightarrow{\varphi} & Y_B.
 \end{array}$$

Here  $f' \circ h = f$ ,  $\psi = (\Delta_B q_\beta) \Delta q_\alpha$ ,  $\varphi$  is the projection of  $\psi(Y)$  onto  $\Delta_B q_\beta(Y)$  ( $\varphi \circ \psi = \Delta_B q_\beta$ ), and  $p'_B \circ h = \Delta_B p_\beta$ . The space  $\bar{X}$  is the fiberwise product of the spaces  $Y$  and  $X_B$  with respect to the mappings  $\varphi \circ \psi$  and  $f_B$  with projections  $p'_B$  and  $f'$ . The mappings  $h$ ,  $f_B$ ,  $\psi$ , and  $\varphi$  are  $n$ -soft, while the mapping  $p'_B \circ h$  is  $(n + 1)$ -soft. We must prove that  $\varphi$  is  $(n - 1)$ -soft. Let  $\bar{Y}$  be the fiberwise product of  $Y_{B \cup \{\alpha\}}$  and  $X_B$  with respect to the mappings  $\varphi$  and  $f_B$ , and let  $\nu: \bar{Y} \rightarrow Y_{B \cup \{\alpha\}}$  and  $\eta: \bar{Y} \rightarrow X_B$  be its projections. Since  $(\psi \circ f'_B) \circ \varphi = f_B \circ p'_B$ , there exists a mapping  $\gamma: \bar{X} \rightarrow \bar{Y}$  such that  $\eta \circ \gamma = p'_B$ , while  $\nu \circ \gamma = \psi \circ f'$ . We show that  $\gamma$  is  $n$ -soft. Suppose that  $\dim Z = n$  and  $A$  is a closed subset of  $Z$ , and let mappings  $\varepsilon: A \rightarrow \bar{X}$  and  $\theta: Z \rightarrow \bar{Y}$  with  $\gamma \circ \varepsilon = \theta|_A$  be given. Since  $\psi$  is  $n$ -soft, there exists a mapping  $\sigma: Z \rightarrow Y$  such that  $\sigma|_A = f' \circ \varepsilon$  and  $\psi \circ \sigma = \nu \circ \theta$ . Since  $(\varphi \circ \psi) \circ \sigma = f_B \circ (\eta \circ \theta)$ , there exists a mapping  $\xi: Z \rightarrow \bar{X}$  such that  $p'_B \circ \xi = \eta \circ \theta$  and  $f' \circ \xi = \sigma$ . Since  $(\psi \circ f') \circ \xi = \nu \circ \theta$ , while  $p'_B \circ \xi = \eta \circ \theta$ , it follows that  $\gamma \circ \xi = \theta$ . Consider the restriction  $\xi|_A$ . It is known that

$$f' \circ \xi|_A = \sigma|_A = f' \circ \varepsilon \quad \text{and} \quad p'_B \circ \xi|_A = \eta \circ \theta|_A = (\eta \circ \gamma) \circ \varepsilon = p'_B \circ \varepsilon.$$

Consequently,  $\xi|_A = \varepsilon$ ; that is,  $\gamma$  is  $n$ -soft. Since  $h$  is  $n$ -soft and  $p'_B \circ h$  is  $(n + 1)$ -soft,  $p'_B$  is  $(n, n + 1)$ -soft. The  $n$ -softness of  $\gamma$  implies the  $(n, n + 1)$ -softness of  $\eta$ . Since  $Y_{B \cup \{\alpha\}} \subset (\prod_B Y_\beta) \times Y_\alpha$  and since  $\varphi$  is the restriction of the projection  $\pi: (\prod_B Y_\beta) \times Y_\alpha \rightarrow \prod_B Y_\beta$ ,  $\varphi$  has a metrizable kernel ( $w(Y_\alpha) \leq \aleph_0$ ). Consequently,  $\eta$  has a metrizable kernel. By Lemma 17,  $\eta$  is  $(n + 1)$ -soft. The mapping  $\varphi$ , being  $n$ -soft, is open; hence it is  $(n + 1)$ -soft by Lemma 16.

The theorem is proved.

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