Homotopy quotients in NCG (a brief sketch)

$A = C^\ast$ algebra describing an NC space
(e.g. a "bad quotient" $X_E$)

$Y$ a commutative space
(only unique up to isopy)

$s.t.$

$k$-theory of $C^\ast$ algebra $A$ (analytic)
can be computed geometrically using
the commut. space $Y$

More precise formulation for group actions: Baum-Connes conjecture

In specific case we're looking at

\[
\rho: K^1(X_E) \to K_0(C^\ast(S(\Lambda,V),\bar{\sigma}))
\]

\[
\rho: K^0(X_E) \to K_1(C^\ast(S(\Lambda,V),\bar{\sigma}))
\]

$\rho$ is "Kasparov map"

Conclusion: The 3-manifold $X_E$ is a homotopy quotient model
of the NC space $C^\ast(S(\Lambda,V),\bar{\sigma}) = A_0 \times V = \frac{\overline{T_0}}{\text{Aut}(T_0)}$

- Spectral geometry on $C^\ast(\Lambda,\sigma)$, $C^\ast(S(\Lambda,V),\bar{\sigma})$
  and on $X_E$

- Relations between these and the Shimizu $L$-function
The Dirac operator on $X_\mathbb{C}$

\[ \mathcal{D}_X = c(\frac{\partial}{\partial t}) \frac{\partial}{\partial x} + c(e^t dx) \frac{\partial}{\partial x} + c(e^t dy) \frac{\partial}{\partial y} \]

because \( \{dt, e^t dx, e^t dy\} \) basis of tangent bundle of \( SL(\mathbb{R}; \mathbb{R}, e) = IR^2 \times IR \)

\( c(w) = \) Clifford multiplication by \( w \)

This gives

\[ \mathcal{D}_X = \frac{\partial}{\partial t} \sigma_0 + e^t \frac{\partial}{\partial x} \sigma_1 + e^t \frac{\partial}{\partial y} \sigma_2 \]

\( \sigma_0, \sigma_1, \sigma_2 = \) Pauli matrices

\[ \mathcal{D}_X = \begin{pmatrix} \frac{\partial}{\partial t} & e^t \frac{\partial}{\partial y} - ie^t \frac{\partial}{\partial x} \\ e^{-t} \frac{\partial}{\partial y} + ie^{-t} \frac{\partial}{\partial x} & -\frac{\partial}{\partial t} \end{pmatrix} \]

Connes–Landi isospectral deformations

\((C^* X, L^2(X, s), \mathcal{D}_X) \) sp. triple of \( * \) span

\( T^2 \subset \text{Isom} (X) \)

\[ \pi(f) = \sum_{n,m \in \mathbb{Z}} f(n,m) \quad f \in C^*(X) \]
Now referring to Shimizu $L$-function
\[ L(N, s) \]

(3 as a Hilbert-Neveu-Singer)

Restrict $D_{0,0}^\mu$ to complement of zero modes $\lambda = 0$

\[ D_{0,0}^\mu = \sum_{\mu \in L(N, s)} \frac{D_{0,0}^\mu}{N(\mu)} \]

Unitarity equiv.

\[ D_{0,0}^\mu \sim D_{0,0}^\mu \frac{1}{Z} \]

\[ D_{0,0}^\mu \chi_\lambda = i \pi^{\nu_{(N, \mu, 1)} N(\mu)^{\frac{\mu \chi}{2}}} \chi_\lambda \]

if $\lambda = A_{\nu}(\mu), \mu \in \nu_{(N, \mu, 1)}$ fundam. domain & $Re \lambda$

\[ D_{0,0}^\mu \chi_\lambda = \left( e^{i \sigma_1} + e^{i \sigma_2} \right) \chi_\lambda \]

\[ \sum_{\mu \in L(N, s)} \frac{\pi^{\nu_{(N, \mu, 1)} N(\mu)^{\frac{\mu}{2}}} = L(N, s) \]

= $\text{Tr} \left( F^0 D_{0,0}^\mu \right)$ Shimizu $L$-function from NC geometry of $A_0$-torus
\[ \mathcal{D}_{\theta,0} = \begin{pmatrix} 0 & \sigma_0 \pm i\sigma_0 \\ \sigma_0 \pm i\sigma_0 & 0 \end{pmatrix} = \lambda \sigma_1 + \lambda \sigma_2 \]

Where \( \lambda = (\lambda_1, \lambda_2) \)
\[ = (\cos \theta, \sin \theta) \]
\[ \psi_x = \psi_{k,m} \]

(Note: part of operator \( \mathcal{D}_x \approx i \frac{\partial}{\partial x} \sigma_3 + 2\pi i \lambda \sigma_1 + 2\pi i \lambda \sigma_2 \)
in Fourier modes along the fiber \( \mathfrak{f}_1 \))

This \( \mathcal{D}_{\theta,0} \) defines a Dirac operator and a spectral triple for the NC torus \( \mathcal{A}_0 \)
with Hilbert space \( \mathcal{H}_0 \cong l^2(\mathbb{Z}) \)

Relation of this spectral triple to spectral geometry on NC space \( C^*(S^1, \mathbb{C}) \):

As seen before \( C^*(S^1, \mathbb{C}) \cong C(\mathbb{R}, \mathbb{C}) \times V \)
\[ = \mathcal{A}_0 \times_{\mathbb{Z}} \mathbb{C} \quad \text{crossed product by } \mathbb{Z} \text{ of an NC-torus} \]

\[ \rightarrow \text{extend a spectral triple on } \mathcal{A}_0 \text{ to a spectral triple on } \mathcal{A}_0 \times_{\mathbb{Z}} \mathbb{C} \]

(work in progress w/ Bellissard-Rehren)

Result in this case: Resulting Dirac op on crossed prod is again \( \mathcal{D}_x \)
\[ [D, \pi(f)] = \sum_{n,m} [D, \pi(f)]_{nm} e^{-2\pi i (\xi_n L_2 + \xi_m L_1)} \]

\[ = \sum_{n,m} [D, \pi(f)]_{nm} e^{-2\pi i (\xi_n L_2 + \xi_m L_1)} \]

Still a Dirac op. (bounded commutators)

\[ \Rightarrow \text{Induced spectral triple from } \mathcal{X}_\Sigma \text{ to NC torus} \]

with real multiplication

\[ A_0 = C^*(\Lambda, \sigma) \]

On standard torus \( \mathbb{R}^2 \) the character functions

\[ s_1 + s_2 \Theta \quad \text{and} \quad \bar{s}_1 + \bar{s}_2 \bar{\Theta} \]

where \( \Theta \in K \) is the Galois conjugate of \( \Theta \)

\[ \mathbb{Z} \Theta \quad \mathbb{Z} \bar{\Theta} \]

pseudolattices in \( \mathbb{R}^2 \)

no lattice \( \Lambda \subset \mathbb{R}^2 \)

\[ \Theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \]

expanding along line \( L_0 = \{s_1 + s_2 \Theta\} \)

contracting along line \( L_0 = \{s_1 + s_2 \bar{\Theta}\} \)

flips all other pts in \( \mathbb{R}^2 \) along hyperbola with asymptotes \( L_0 \) & \( L_0 \)

\[ \frac{e^t \partial}{\partial t} + \frac{e^{-t} \partial}{\partial y} \]

in \( \mathcal{X}_\Sigma \) are the leafwise derivatives along these two foliations

with \( e^t \) & \( e^{-t} \) factors

stability of transverse measure under the flow
\[ \alpha \tau (\pi (f_{n,m})) = e^{2\pi i (n \tau_1 + m \tau_2)} \pi (f_{n,m}) \]

\[ \tau = (\tau_1, \tau_2) \in T^2 \]

\[ \alpha \tau (T) = U(\tau) T U(\tau)^* \]

\[ \forall \tau \in T^2 \]

\[ \forall \tau \in T^2 \]

\[ U(\tau) \text{ unitary transformation } T^2 \text{ act on } \mathcal{H} \]

\[ U(\tau) = \exp (2\pi i \tau L) \quad L = (L_1, L_2) \text{ infinitesimal generator of this action} \]

\[ \Rightarrow \quad \pi (f_{n,m}, \tau) \mathcal{T}_{\tilde{\tau}_1, \tilde{\tau}_2} f = \sum_{n,m} \pi (f_{n,m}) e^{-2\pi i (\frac{n\tau_1}{2} \tilde{\tau}_1 + \frac{m\tau_2}{2} \tilde{\tau}_2)} f_{n,m} \mathcal{T}_{\tilde{\tau}_1, \tilde{\tau}_2} \]

\[ \pi (f_{n,m}, \tau) \mathcal{T}_{\tilde{\tau}_1, \tilde{\tau}_2} f_{n,m} = \pi \mathcal{T}_{\tilde{\tau}_1, \tilde{\tau}_2} f_{n,m} + h_{k,r} \]

\[ f_{n,m} \ast h_{k,r} = \exp \left( -2\pi i \left( \frac{n}{2} \tau_1 + \frac{k}{2} \tau_2 \right) \right) f_{n,m} \ast h_{k,r} \]

\[ \Rightarrow \text{ NC torus with } \sigma ((n,m), (k,r)) = \exp \left( -2\pi i \left( \frac{n}{2} \tau_1 + \frac{k}{2} \tau_2 \right) \right) \]

Geometrically: take \( X_\varepsilon : \text{O} \xrightarrow{\varepsilon} T^2 \)

\[ \text{fibration of } \text{tori over } S^1 \]

and replace with

\[ \text{fibration of NC torus over } S^1 \]

Same Hilbert space \( \mathcal{H} \) and same Dirac operator

\[ U(\tau) D U(\tau)^* = D \]

since \( T^2 \) acts on \( X \) by isometries