Bost-Connes algebra

First description: generators and relations

$\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ group ring

$\mathbb{Q}/\mathbb{Z}$ additive group $\mathbb{Q}$ mod $\mathbb{Z}$

$r \in \mathbb{Q}/\mathbb{Z}$ basis elements δ_r of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

$\Sigma\, a_r \delta_r$ finite sums

where $\delta_r \delta_s = \delta_{r+s}$

use notation $e(r)$ for these generators δ_r

reason: what to "suggest"... exponential map

$$e(r+s) = e(r)\, e(s)$$

$$e(0) = 1$$

~~[illegible]~~ Additional generators μ_n $n \in \mathbb{N}$, $n = 1, 2, 3, \dots$

and "adjoints" μ_n^*

with relations

$$\left\{\begin{array}{l} \mu_n^* \mu_n = 1 \\ \mu_n \mu_n^* = e_n \quad \text{an idempotent in } \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \\ \mu_{nm} = \mu_n \mu_m \\ \mu_n \mu_m^* = \mu_m^* \mu_n \ \text{ if } (m,n) = 1 \end{array}\right.$$

$$e_n = \frac{1}{n} \sum_{s:\, ns = 0} e(s)$$

[need to verify: $e_n^2 = e_n$ ($= e_n^*$)]

$$\mu_n\, e(r)\, \mu_n^* = \frac{1}{n} \sum_{ns = r} e(s)$$

Algebra defined over $\mathbb{Q}$ $\qquad \mathcal{A}_{BC,\mathbb{Q}}$

(over $\mathbb{C}$: $\quad \mathcal{A}_{BC,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \mathcal{A}_{BC,\mathbb{C}} = \mathbb{C}[\mathbb{Q}/\mathbb{Z}] \rtimes_{\rho} \mathbb{N}$

C^*-completion $\quad C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_{\rho} \mathbb{N}$

Note : $C^*(\mathbb{Q}/\mathbb{Z})$ group C^*-alg. of abelian group

$\Rightarrow \quad C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\widehat{\mathbb{Q}/\mathbb{Z}})$ isom. by Fourier transform to alg of cont. funct. on Pontrjagin dual group

$$\left.\begin{array}{l} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, U(1)) \\ \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{C}^*) \end{array}\right\}$$

a homom. is going to map $\mathbb{Q}/\mathbb{Z}$ to roots of 1 in $\mathbb{C}^*$ (in $U(1)$) because all torsion pts

= All ways of mapping by group homomorphism $\mathbb{Q}/\mathbb{Z}$ to roots of 1

$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \qquad = \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} \qquad =: \hat{\mathbb{Z}}$ or (the meaning profin. completion)

m/n ordered by

pairing $\langle \gamma, \rho \rangle = \exp(2\pi i \rho(\gamma))$ $\gamma \in \mathbb{Q}/\mathbb{Z}$

$\Rightarrow \quad C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}})$

Bost-Connes algebra also described as

$$C(\hat{\mathbb{Z}}) \rtimes_{\rho} \mathbb{N}$$

Semigroup crossed product:

· The μ_n's alone generate semigroup ring
$\mathbb{C}[\mathbb{N}]$

$\mathbb{N}$ = multiplicative abelian semigroup
generated by primes

$\{\mu_p\}_{p = \text{primes}}$ generators

$\mu_p^* \mu_p = 1 \qquad \mu_p \mu_p^* = e_p$ projector

$\Rightarrow$ Toeplitz algebra on generators μ_p

$\bigotimes_p \mathcal{T}_p$
(toeplitz alg on single μ_p generator

Semigroup crossed product:

A = abelian alg. $A = \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ or $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ here

$\forall n \in \mathbb{N}$ $\rightsquigarrow$ endomorphisms

$\rho_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ endom.

$$\rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s)$$

(<u>note</u>: sends 1 to e_n not to 1)

isom: from $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ to reduced alg. by e_n: $e_n A e_n = e_n A$

$\rho_n(e(r)) = e_n\, e(s) \quad \forall s$ s.t. $ns = r$

Also $\rho_n(e(r)) = \mu_n e(r) \mu_n^*$

Have also other family of endom. of $A = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ from semigroup $\mathbb{N}$

$$\sigma_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \to \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$$

$$\sigma(e(r)) = e(nr) \qquad \forall n \in \mathbb{N} \quad \forall r \in \mathbb{Q}/\mathbb{Z}$$

These satisfy:

$$\begin{cases} \sigma_n \rho_n(x) = x \\ \rho_n \sigma_n(x) = e_n \cdot x \end{cases}$$

If had a group instead of semigroup action would be implemented by unitaries U_γ not isometries μ_s and would have isomorphisms $U_\gamma a U_\gamma^*$ and $U_\gamma^* a U_\gamma$ instead of endom.

Note: multiplication by n

$n : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is surjective (divisibility of $\mathbb{Q}/\mathbb{Z}$)

Note: ρ_n's surjective onto $e_n A$ (not onto A!) not inj.
σ_n's injective (on $e_n A$)

$$\sigma_n(e(r)) = \mu_n^* e(r) \mu_n$$

Look at representations: 5

$\mathcal{H} = \ell^2(\mathbb{N})$ on. basis $\{\varepsilon_n\}$

$$\mu_n \varepsilon_m = \varepsilon_{nm}$$

$$\mu_n^* \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \varepsilon_{\frac{m}{n}} & n \mid m \end{cases} \qquad \mu_n^* \mu_n = 1$$

$$\mu_n \mu_n^* = \text{Projection on } \{\varepsilon_m : n \mid m\}$$

$e(r)$: choose an embedding of roots of unity into $\mathbb{C}$ by group homom.

$$\rho \in \hat{\mathbb{Z}}^* = GL_1(\hat{\mathbb{Z}})$$

$$\Rightarrow \quad \pi_\rho(e(r))\, \varepsilon_n = \zeta_r^n \varepsilon_n$$

$\zeta_r = \rho(e(r))$ root of 1 : $\zeta_r = (\zeta_b)^a$
$r = \frac{a}{b}$ (prim. b-th root of 1

then all relations satisfied

need to check crossed prod. relation

$$\mu_n \pi(e(r)) \mu_n^* \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \zeta_r^{\frac{m}{n}} \cdot \varepsilon_m & n \mid m \end{cases}$$

but notice that

$e_n = \frac{1}{n} \sum_{sn=0} e(s)$ acts as

$$\frac{1}{n}\sum_{ns=r} e(s) \quad \varepsilon_m = \frac{1}{n}\sum_{ns=r} \zeta_s^m \quad \varepsilon_m$$

$s = \frac{a}{b}$ $\quad ns = 0 \quad b/n \quad r = \frac{c}{d}$

$\zeta_s = \zeta_b^a \qquad \frac{na}{b} = \frac{c}{d} \qquad \frac{n}{b} = \frac{1}{d}$

- if $n \nmid m$ summing over all roots of 1 (ζ) of order $\frac{b}{n} = d$ $\Rightarrow$ zero
- if $n \mid m$ summing 1 n times with $\frac{1}{n}$ $\Rightarrow$ one

result

$0 \quad n \nmid m$
$1 \quad n \mid m$ } same as projection above $e_n = e_n^2 = e_n^*$

Time evolution (Bost-Connes system)

$$\begin{cases} \sigma_t(\mu_n) = n^{it}\mu_n \\ \sigma_t(e(r)) = e(r) \end{cases}$$

$(\, C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\rho \mathbb{N} \,,\ \sigma_t\,)$ quantum statistical mechanical system

In all representation π_ρ $\rho \in \hat{\mathbb{Z}}^*$

on $\mathcal{H} = \ell^2(\mathbb{N})$)

Hamiltonian of time evolution:

$$\pi_\rho\left(\sigma_t(a)\right) = e^{itH}\, \pi_\rho(a)\, e^{-itH}$$

check for μ_n

$$\underbrace{\pi_\rho(\mu_n)}_{\text{indep of } \rho}\ \varepsilon_m = \varepsilon_{nm}$$

$$\pi_\rho\left(\sigma_t(\mu_n)\right)\varepsilon_m = n^{it}\, \varepsilon_{nm}$$

$$= (nm)^{it}\, m^{-it}\, \pi_\rho(\mu_n)\, \varepsilon_m$$

get: $\quad e^{itH}\varepsilon_m = m^{it}\,\varepsilon_m$

$$H\,\varepsilon_m = \log m\ \varepsilon_m$$

Hamiltonian (up to constant)

$$\Rightarrow \quad Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{n \geq 1} n^{-\beta} = \zeta(\beta)$$

Partition function is the Riemann zeta function

Symmetries of the system:

$\hat{\mathbb{Z}}^*$ acts by automorphisms of the algebra

$\alpha \in \hat{\mathbb{Z}}^*$ $\quad \alpha(\mu_n) = \mu_n$

$\alpha(e(r)) = e(\alpha(r))$

since $\alpha \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ invertible

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Before describing classification of KMS states for $(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\rho \mathbb{N}, \sigma_t)$

A different more geometric description of the algebra:

<u>$\mathbb{Q}$-lattices and commensurability</u>

$\mathbb{Q}$-lattice in n-dimensions

$\Lambda \subset \mathbb{R}^n$ lattice (abelian group, abstractly isomorphic to $\mathbb{Z}^n$, embedded in $\mathbb{R}^n$ with compact quotient $\mathbb{R}^n/\Lambda \simeq \underbrace{T^n}_{\text{torus}} \simeq (S^1)^n$)

together with

ϕ : $\mathbb{Q}^n/\mathbb{Z}^n \xrightarrow{\phi} \mathbb{Q}\Lambda/\Lambda$

a <u>group homomorphism</u> (not required to be an isomorphism)

A $\mathbb{Q}$-lattice (Λ, ϕ) is "invertible" if ϕ is an isomorphism

Equivalence relation on $\mathbb{Q}$-lattices

Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$

iff $\Lambda_1 \mathbb{Q} = \Lambda_2 \mathbb{Q}$ i.e. the two lattices are commensurable

and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

Note. $\Lambda_1 + \Lambda_2$ is a lattice because Λ_1 & Λ_2 commensurable (in general not)

It is an equivalence relation:

$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ and $(\Lambda_2, \phi_2) \sim (\Lambda_3, \phi_3)$

then also $(\Lambda_1, \phi_1) \sim (\Lambda_3, \phi_3)$

- the lattices Λ_i of finite index in $\Lambda_1 + \Lambda_2 + \Lambda_3$

$$\# (\Lambda_1 + \Lambda_2 + \Lambda_3) / \Lambda_i < \infty$$

$$\phi_1 - \phi_2 = 0 \mod \Lambda_1 + \Lambda_2 \Rightarrow \mod \Lambda_1 + \Lambda_2 + \Lambda_3$$
$$\phi_2 - \phi_3 = 0 \mod \Lambda_2 + \Lambda_3 \Rightarrow \mod \Lambda_1 + \Lambda_2 + \Lambda_3$$

$\Rightarrow \phi_1 - \phi_3 = 0 \mod \Lambda_1 + \Lambda_2 + \Lambda_3$

want $\mod \Lambda_1 + \Lambda_3$

induced homom. $\phi_1 - \phi_3 : \underbrace{\mathbb{Q}^n / \mathbb{Z}^n}_{\text{infinitely divisible}} \longrightarrow \underbrace{(\Lambda_1 + \Lambda_2 + \Lambda_3) / (\Lambda_1 + \Lambda_3)}_{\text{finite}}$

$\Rightarrow$ group homom. trivial $\Rightarrow \phi_1 - \phi_3 = 0 \mod \Lambda_1 + \Lambda_3$

Case of 1-dimensional $\mathbb{Q}$-lattices:

Note: in general commens. relation on $\mathbb{Q}$-lattices "bad" equiv. rel. quotient NC space

$\Lambda \subset \mathbb{R}$ 1-dimensional

is a scaled copy of $\mathbb{Z}$

$\Lambda = \lambda \mathbb{Z}$ for some $\lambda \in \mathbb{R}^*_+$

$\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}\Lambda/\Lambda$ group homom.

after identifying $\Lambda = \lambda\mathbb{Z}$ ϕ completely determined by an element

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$

$$\phi = \lambda\rho$$

1-dimensional $\mathbb{Q}$-lattices <u>up to scaling</u>

forget the factor $\lambda \in \mathbb{R}^*_+$

$\Rightarrow$ just $\mathbb{Z}$ with an element $\rho \in \hat{\mathbb{Z}}$

algebra of functions $C(\hat{\mathbb{Z}})$ $\left(\cong C^*(\mathbb{Q}/\mathbb{Z})\right)$

<u>Commensurability relation</u>:

rescaling lattice by $\frac{n}{m}\Lambda$ $\frac{n}{m}\mathbb{Z}$

$r = \frac{n}{m} \in \mathbb{Q}^*_+$

up to scaling $\rho_2 = r\rho_1$

$\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

$m\rho \in \hat{\mathbb{Z}}$

pairs of commens. lattices (ρ_1, λ_1) (ρ_2, λ_2) $\exists r \in \mathbb{Q}^*_+$ $\rho_2 = r\rho_1$ $\lambda_2 = r\lambda_1$

Groupoid of equiv. rel. (11)

$$\mathcal{G} = \{(r, \rho, \lambda) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} \times \mathbb{R}_+^* \ : \ r\rho \in \hat{\mathbb{Z}}\}$$

source, range maps

$$(r, \rho, \lambda) \longmapsto (\rho, \lambda)$$

$$(r, \rho, \lambda) \longmapsto (r\rho, r\lambda) \qquad \text{composition}$$

$$(r_1, \rho_1, \lambda_1) \circ (r_2, \rho_2, \lambda_2) = (r_1 r_2, \rho_2, \lambda_2)$$

$$\text{if} \quad r_2 \rho_2 = \rho_1 \qquad r_2 \lambda_2 = \lambda_1$$

Groupoid algebra $C^*(\mathcal{G}_1 / \mathbb{R}_+^*)$

quotient of groupoid $\mathcal{G}_1$ by scaling action (still groupoid)

$$(r_1 \rho_1) \circ (r_2, \rho_2) = (r_1 r_2, \rho_2) \qquad r_2 \rho_2 = \rho_1$$

$$\begin{cases} (f_1 * f_2)(\gamma) = \sum_{\gamma = \gamma_1 \circ \gamma_2} f_1(\gamma_1) f_2(\gamma_2) \\ f^*(\gamma) = \overline{f(\gamma^{-1})} \end{cases}$$

This algebra is the same as Bost-Connes algebra

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\rho \mathbb{N}$$

Time evol. by covolume of commensurable lattices

$$\sigma_t(f)(r, \rho) = r^{it} f(r, \rho) = \left(\frac{\mathrm{Covol}(\Lambda_1)}{\mathrm{Covol}(\Lambda_2)}\right)^{it} f((\Lambda_1, \phi_1), (\Lambda_2, \phi_2))$$

$n\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$ range of multiplication

(open and closed)

$e_n = \chi_{n\hat{\mathbb{Z}}}$ characteristic function $\in C(\hat{\mathbb{Z}})$

$e_n e_m = e_{\mathrm{lcm}(n,m)}$

$e_n = \chi_{\{\mathbb{Q}\text{-lattices divisible by } n\}}$

(Λ, ϕ) divisible by N
if $\phi_N : (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^r \to \frac{1}{N}\Lambda/\Lambda$
is $\phi_N = 0$.

Semigroup action

$\alpha_n(f)(\Lambda, \phi) = f(n\Lambda, \phi)$ — if $(\Lambda,\phi) \in \mathrm{supp}(e_n)$; <u>commensurability relation</u>; zero otherwise

same as

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \rho \in n\hat{\mathbb{Z}} \\ 0 & \rho \notin n\hat{\mathbb{Z}} \end{cases}$$

$$\mu_n(r, \rho) = \begin{cases} 0 & r \neq n \\ 1 & r = n \end{cases}$$

$(r, \rho) \leftrightarrow ((r^{-1}\mathbb{Z}, \rho), (\mathbb{Z}, \rho))$
pair of commens. $\mathbb{Q}$-lattices

$\mu_n * f * \mu_n^* = \alpha_n(f)$

convol. prod in $C^*(\mathcal{G}_1/\mathbb{R}_+^*)$