EZADS Inputs which Produce Half-Factorial Block Monoids

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1 Introduction and definitions

In this talk, I will be presenting the research I did at the 2010 Mathematics REU at the University of Minnesota, Duluth. I studied the factorization properties of a special class of block monoids called EZADS (Erdös-Zaks All Divisor Sets) monoids. I will first define block monoids and introduce the notion of half-factorality (a waeker version of unique factorization). I will then summerize the EZADS contruction, which is inspired by a paper by Erdös and Zaks, and describe my main results about EZADS monoids.

Let G be an abelian group, and let S be a nonempty subset of G. Let $\mathcal{F}(S)$ be the free abelian monoid generated by S, so that any element $B \in \mathcal{F}(S)$ can be written uniquely in the form $B = \prod_{s\in S} s^{v_s}$, where each v_s is a nonnegative integer. Consider the map $\pi : \mathcal{F}(S) \to G$ defined by $\pi\left(\prod_{s\in S} s^{v_s}\right) = \sum_{s\in S} v_s s$. Let $\mathcal{B}(G,S) = \ker \pi$. This is known as *the block monoid of G with respect to S*, and the elements of $\mathcal{B}(G,S)$ are known as *blocks*. One may think of $\mathcal{B}(G,S)$ as the monoid of 'relations' on the set S.

We may now define the notion of divisibility in $\mathcal{B}(G,S)$ in the standard way. Namely for $B, C \in \mathcal{B}(G,S)$, we say that C *divides* B, and write $C|B$ if there exists some $D \in \mathcal{B}(G,S)$ for which $CD = B$. A block B is said to be *irreducible* (or an *atom*) if its only divisors are itself and the empty block. Let $\mathcal{A}(\mathcal{B}(G,S))$ be the set of irreducible blocks.

We say that $\mathcal{B}(G,S)$ is *half-factorial* if for any $B \in B(G,S)$, there is a unique integer n for which B can be written as the product of n irreducibles. In other words, $\mathcal{B}(G,S)$ has factorization into a unique *number* of irreducibles. This is a strictly weaker condition than unique factorization. We also say that S is a *half-factorial subset* of G.

From now on we will assume that G is finite. For a block $B = \prod_{s \in S} s^{v_s}$, define the *cross number* by

$$\Bbbk(B) = \sum_{s \in S} \frac{v_s}{|s|}.$$

The cross numbers of irreducible blocks can be used to determine whether $\mathcal{B}(G,S)$ is half-factorial. Specifically we have the following theorem.

Theorem 1.1. *If G is torsion and $S \subseteq G$ with $S \neq \varnothing$ then $\mathcal{B}(G,S)$ is half-factorial if and only if $\Bbbk(B) = 1$ for every $B \in \mathcal{A}(\mathcal{B}(G,S))$.*

In the general case, Theorem 1.1 is difficult to apply. First,the cross numbers of $\mathcal{B}(G,S)$ can be hard to understand. Moreover the set $\mathcal{A}(\mathcal{B}(G,S))$ is often quite difficult to describe.

These concerns inspired the EZADS construction. Given a set of pairwise relatively prime positive integers $I = \{a_1, \ldots, a_n\}$ (called an *EZADS input*) let $q = a_1 \cdots a_n$ and $q_k = q/a_k$ for all k. Consider the block monoid $\mathcal{B}(I) := \mathcal{B}(\mathbb{Z}/q\mathbb{Z}, \{1, q_1, \ldots, q_n\})$.

The irreducible elements of $\mathcal{B}(I)$ turn out to be relatively easy to describe. In fact, one can construct a sequence of blocks $\{\mathfrak{M}_i\}_{i=1}^{q-1}$ in $\mathcal{B}(I)$ which is gaurenteed to contain all irreducibles. Moreover, the cross number of any block in $\mathcal{B}(I)$ is an integer. This makes applying Theorem 1.1 much easier than in the general case.

The primary goal of my research was to determine for which EZADS inputs, I, the resulting block monoid $\mathcal{B}(I)$ is half-factorial.

I will first describe my initial results. Then I will show how these results can be used to reformulate the problem in terms of real-valued quantities. I will then apply these results to the case where $n = 3$. I will describe a finite algorithm which, for any fixed m, will output a complete classification of EZADS inputs of the form $\{m, a, b\}$ which produce a half-factorial block monoid.