

From Sets to Categories

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the Zermelo-Fraenkel axioms describe objects which they call “sets”, but which we would *not* call sets but rather “*trees with no symmetries*” (Vladimir Voevodsky)

the substance of mathematics resides not in Substance (as it is made to seem when \in is the irreducible predicate, with the accompanying necessity of defining all concepts in terms of a *rigid elementhood relation*) but in Form (as is clear when the guiding notion is *isomorphism-invariant structure*, as defined, for example, by *universal mapping properties*) (F. William Lawvere)

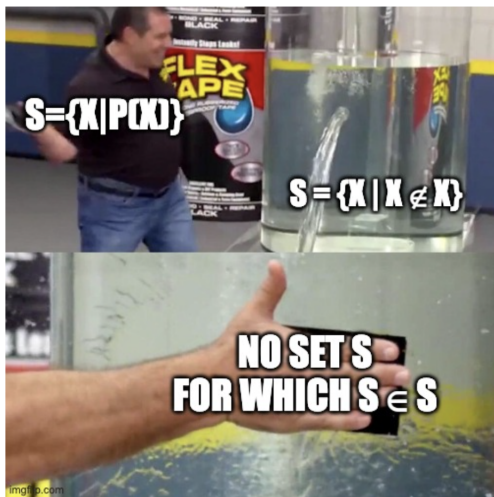
Russell paradox

- define sets by **properties**: $y = \{x \mid P(x)\}$
- take as property $P(x)$ the formula $x \notin x$
- then consider **set** $y = \{x \mid x \notin x\}$: is $y \in y$?
- if yes, then $y \notin y$, if no then $y \notin y$ so $y \in \{x \mid x \notin x\} = y$
- Russell paradox (again it looks like the Epimenides paradox: “this sentence is a lie”)
- answer: **adding axiom** there is no set x with $x \in x$
- in ZF theory: **axiom of regularity**:

$$\forall x(x \neq \emptyset \rightarrow (\exists y \in x)(x \cap y = \emptyset))$$

- consequence: no sets contain themselves as elements: x set, take $\{x\}$ by regularity exists element (only x) with $x \cap \{x\} = \emptyset$, so element $x \in \{x\}$ cannot be element $x \in x$

ZF set theory and the Russell paradox:



...but the actual problem is with \in itself (membership)

What is a set? (heuristic discussion)

earlier view:

- a “set” is a collection **determined by the members** it contains
- ways of constructing new sets given other sets (unions, intersections, power sets, pairing, etc)
- *members of a set are other sets* (!) so in particular members (elements) of a set have “**inner structure**” (they can themselves have elements, etc)
- so membership \in is a relation on some “set of sets” (!)

modern view:

- elements $x \in X$ of a set are *not* sets (rather maps $x : \{\star\} \rightarrow X$)
- elements of X are not defined independently of X (so two sets cannot share elements, unless they are subsets of some fixed set)
- a set X is **determined uniquely by its “extended elements”** aka maps $f : Y \rightarrow X$ from other sets Y (Yoneda)
- constructing new sets (products, coproducts, etc): universal properties

Zermelo-Fraenkel theory of sets

- first-order language $\mathcal{L} = \{\in\}$ binary relation of membership \in
- 8 axioms of theory ZF; theory ZFC: augmented with C=(the “axiom of choice”)

1. *Axiom of Extensionality*

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Sets are equal if they have the same elements.

2. *Axiom of Pairing*

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

For each x and y , $\{x, y\}$ is a set.

3. *Axiom Scheme of Separation*

For every formula ϕ not containing the variable y , we have an axiom

$$\forall x \exists y \forall w (w \in y \leftrightarrow (w \in x \wedge \phi))$$

For each set x and property ϕ , $\{w \in x \mid \phi\}$ is a set.

4. *Axiom of Union*

$$\forall x \exists y \forall w (w \in y \leftrightarrow \exists z (z \in x \wedge w \in z))$$

For every set x , $\bigcup x$ (or $\bigcup_{z \in x} z$) is a set.

5. *Axiom of Power Set*

$$\forall x \exists y \forall w (w \in y \leftrightarrow \forall z (z \in w \rightarrow z \in x))$$

For every set x , $\mathcal{P}(x)$ is a set.

comments on this part of the axioms:

- establishing **membership** as the way to determine sets
- providing a basic operation of “**set formation**” via pairing (sometimes also called “set theoretic merge”): *not* a union! $\{x, y\}$ has the sets x and y as its *elements* (the elements of x and y are no longer elements of the resulting set)
- establishing **properties** as a main way of constructing sets
- forming **unions** of sets indexed by other sets

$$Y = \bigcup_{x \in X} Y_x$$

- providing a construction for the “**set of all subsets**” $\mathcal{P}(x)$ of a given set x (**power set**)
- **main set constructions**: pairing, union, properties, power set

6. *Axiom of Infinity*

Since $\exists x(x = x)$ is a valid sentence, there is a set. If x is a set then by Separation there is a set $\{w \in x \mid \perp\}$ which has no elements; and this set is unique by Extensionality. We denote this empty set by \emptyset . Also, for any set x we have a set $\{x\} = \{x, x\}$ by Pairing, and $x \cup \{x\} = \bigcup\{x, \{x\}\}$ using Pairing and Union. With these notations, the axiom of Infinity is now

$$\exists x(\emptyset \in x \wedge \forall y(y \in x \rightarrow (y \cup \{y\}) \in x)).$$

Suppose x is such a set as postulated by the axiom of Infinity. Then $\emptyset \in x$; let us write 0 for \emptyset . Then, also $0 \cup \{0\} \in x$; we write 1 for this set. Inductively, we write $n + 1$ for $n \cup \{n\}$. We can prove that n has exactly n elements, and therefore, that the set x “is infinite” in the sense that we can write down infinitely many elements of x (which are pairwise distinct).

so ZF contains von Neumann construction of integers \mathbb{N}

7. Axiom Scheme of Replacement

For any formula ϕ which does not contain the variable y :

$$\forall a \exists b \forall c (\phi(a, c) \leftrightarrow c = b) \rightarrow \\ \forall x \exists y \forall v (v \in x \rightarrow \exists u (u \in y \wedge \phi(v, u)))$$

the first line says that ϕ defines an operation F on sets and the axiom (“axiom scheme” as first-order logic does not quantify over ϕ so one axiom for each such ϕ) is that for any such operation and any set x one obtains a set $\{F(v) \mid v \in x\}$

8. Axiom of Regularity

$$\forall x (x \neq \emptyset \rightarrow \exists y (y \in x \wedge \forall z \neg (z \in y \wedge z \in x)))$$

Every nonempty set x has an element that is disjoint from x .

So no “set of all sets” and no Russell paradox as (by axiom) $\{x \mid x \notin x\}$ would have to be “set of all sets”

ZFC = ZF augmented by the axiom of choice

- **abbreviations:** write $\exists x \in y \phi$ for $\exists x(x \in y \wedge \phi)$ and $\forall x \in y \phi$ for $\forall x(x \in y \rightarrow \phi)$
- **ordered pairs:** (x, y) defined as $\{\{x\}, \{x, y\}\}$
- **product:** $x \times y = \{(u, v) \mid u \in x \wedge v \in y\}$ (is a set: (u, v) subset of $\mathcal{P}(x \cup y)$)
- **relation** R subset of $x \times y$, R is a **function** if $\forall u \in x \exists v \in y \forall w \in y ((u, w) \in R \leftrightarrow v = w)$
- there is a “set of all functions” from x to y , denote by y^x

Axiom of Choice:

$$\forall x \forall y \forall f \in y^x [\forall v \in y \exists u \in x ((u, v) \in f) \rightarrow \exists g \in x^y \forall v \in y \forall u \in x ((v, u) \in g \rightarrow (u, v) \in f)].$$

existence of a **section** for any $f : x \rightarrow y$

are the objects described by ZF axioms actually sets?

- empty set $\emptyset = \{v \in x \mid \perp\}$, unique by Extensionality, OK
- sets with one element: $\{x\}$ for x any set (all these different sets for different x)

$$\{\{x, y\}\} \neq \{x\} \neq \{y\}$$

- but there is really only **one** set with one element (unique up to unique isomorphism): one is simply *relabeling* that single element with a different name...
- the situation gets worse with larger sets: with two elements how is $\{0, 1\}$ different from any $\{a, b\}$ besides a name change? in ZF the **inner structure** of a, b (and of $0 = \emptyset$ and $1 = \{\emptyset\}$) makes them actually different
- **why should elements of sets have an inner structure?**
- in the view of *modern mathematics* this makes them something **other** than sets

another view on ZF: *they are not sets but rooted trees!*

- ϵ means “child of”
- $\emptyset =$ single root
- **pairing** is binary tree Merge

$$\begin{array}{c} \wedge \\ x \quad y \end{array} = \{x, y\}$$

takes two rooted trees and appends the two roots to a common root

- Note: the trees are *not planar!*
- now for example $\{x\}$ and $\{y\}$ that should be *indistinguishable* as sets are indeed *different* as trees
- this pairing operation only produces binary branchings, unary $x \mapsto \{x\}$ produces non-branching vertices (**important**: this unary operation exists in ZF because pairing $\{x, x\} = \{x\}$)

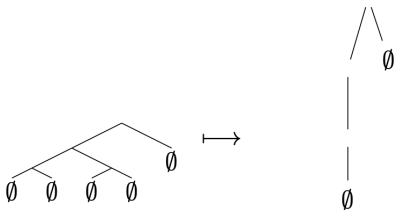
Important restriction: ZF only deals with

trees with NO SYMMETRIES!

- because $\{x, x\} = \{x\}$
- where by a *symmetry* in a rooted tree T we mean a non-leaf node v such that the subtree $T_v \subseteq T$ rooted at v is of the form

$$T_v = \begin{array}{c} \wedge \\ T' \quad T' \end{array}$$

Example: first tree is *not* in ZF but second one is: obtained from first by imposing relation $\{x, x\} = \{x\}$



other ZF axioms

- **extensionality**: identity of trees $T = T'$ verified recursively by $\mathcal{B}^-(T) = \mathcal{B}^-(T')$
- **separation** in ZF is a **pruning of trees** operation: only keep those branches of tree where property ϕ holds
- **power set axiom**:
 - 1 cut tree T below the roots (get a forest)
 $\mathcal{B}^-(T) = F = T_1 \sqcup \dots \sqcup T_k,$
 - 2 take all bi-partitions of F : pairs of $F_J = \sqcup_{j \in J} T_j$ and $F_{J^c} = \sqcup_{j \in J^c} T_j$ for all $J \subseteq \{1, \dots, k\}$
 - 3 apply $\mathcal{B}^+(\sqcup_J \mathcal{B}^+(F_J))$

that's $x \mapsto \mathcal{P}(x)$ in tree-form

- **Regularity**: the trees are indeed trees (no loops)

Note: union axiom allows for infinite branching (hence \mathcal{B}^- can produce forests with infinitely many components)

But then what are sets? (first heuristic discussion)

- change of perspective on **membership**
- morphisms $f : A \rightarrow B$ (functions between sets) can be thought of as “elements of the set B over the domain of variation A ” (like “elements of B that depend on parameters in A ”)
- in particular if $A = \{\star\}$ (*the* set with one element - unique up to unique isomorphism) then a function $x : \{\star\} \rightarrow B$ is just an element of B (so $x : \{\star\} \rightarrow B$ replaces $x \in B$)
- elements are entirely “relational” (functions), and do not exist on their own outside of B
- **cannot compare sets through their elements!** only through maps
- when emphasis is on maps and relational notions, it means the right mathematical framework is **category theory**

- example of relational ways of thinking of operations on sets:
intersection
- since different sets A, B cannot have the “same” elements (if elements are maps), what is $A \cap B$? **does not make sense intrinsically**
- the only way maps $x : \{\star\} \rightarrow A$ and $y : \{\star\} \rightarrow B$ can be compared is if they both end up in the same place... after composition of morphisms
- given a diagram

$$\begin{array}{ccc}
 & & A \\
 & & \downarrow j_A \\
 B & \longrightarrow & C
 \end{array}$$

j_B

with $j_A : A \rightarrow C$ and $j_B : B \rightarrow C$ two injective maps, then one can ask for completion of diagram that is **optimal** (i.e. a solution to a universal property: the fibered product)

- this is **intersection** of A and B as mapped inside C by j_A, j_B

Categories (a first look)

- a category \mathcal{C} consists of
 - 1 a **collection** of objects $\text{Obj}(\mathcal{C})$
 - 2 a **collection** of morphisms $\text{Hom}_{\mathcal{C}}$
- with properties
 - each morphism f has **source and target** objects $X = s(f)$ and $Y = t(f)$ so morphisms subdivided into $\text{Hom}_{\mathcal{C}}(X, Y)$
 - for each object X there is an **identity morphism** 1_X
 - **composition** $h = g \circ f$ of morphisms $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, Z)$

$$\circ : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z)$$

- composition is **associative** $(f_1 \circ f_1) \circ f_3 = f_1 \circ (f_2 \circ f_3)$
- **units**: $f \circ 1_{s(f)} = 1_{t(f)} \circ f = f$

What about “**collection**” (**classes**)? don't want to use **set** here, though in many cases (small categories) these will be sets

Quick note on proper classes:

- how to treat $\text{Obj}(\mathcal{C})$ when *not* a set? (e.g. $\text{Obj}(\text{Sets})$)
- **classes**: notion of **collection** of objects sharing some common properties
- **proper classes**: classes that are not sets
- **idea**: “class builder expressions” $\{x \mid \phi\}$ with ϕ logical formula
- two formulae ϕ, ψ define same class if equivalent $\phi \leftrightarrow \psi$ (so think of classes as equivalence classes of formulae)
- in ZFC sets can be elements while proper classes cannot
- in theories like ETCS classes represented by first-order formulae with a free variable: a proper class is a formula that cannot be represented by a set (an object of the category)

- another approach to *proper classes*: **inaccessible cardinals** (cardinal number is an isomorphism class of sets, like cardinality for finite sets)
- **very rough sketch of idea**:
- “inaccessible cardinal” κ : cardinal that is “regular” (any sum of $< \kappa$ cardinals each $< \kappa$ is also $< \kappa$, sum is cardinal of disjoint union) “strong limit” (if $\lambda < \kappa$ then $2^\lambda < \kappa$)
- so “inaccessible” because set theoretic constructions (union, power-set) on lower cardinals does not reach κ
- example \aleph_0 inaccessible from just finite sets
- require also that inaccessible cardinals κ is *uncountable*
- **universe** $\mathcal{V}_\kappa = \bigcup_{\lambda < \kappa} \mathcal{V}_\lambda$ union of all sets of cardinality $< \kappa$
- **proper classes**: subsets $C \subseteq \mathcal{V}_\kappa$ that are not elements $C \in \mathcal{V}_\kappa$
- existence of inaccessible cardinals is taken as a postulate

Functors and Natural Transformations

- **Functors** $F : \mathcal{C} \rightarrow \mathcal{C}'$, on objects $\mathcal{O}(\mathcal{C}) \ni X \mapsto F(X) \in \mathcal{O}(\mathcal{C}')$, on morphisms $F(f) : F(X) \rightarrow F(Y)$ (covariant; also contravariant) with $F(g \circ f) = F(g) \circ F(f)$ and $F(1_X) = 1_Y$
- **Natural Transformations:** $\eta : F \rightarrow G$, to every object a morphism $\eta_X : F(X) \rightarrow G(X)$ with $\eta_Y \circ F(f) = G(f) \circ \eta_X$

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

common use example: finite directed graphs as functors



- category $\mathbf{2}$ with two objects V, E and two non-identity morphisms $s, t : E \rightarrow V$
- functor $G : \mathbf{2} \rightarrow \mathbf{FinSets}$, sets V_G and E_G (vertices, edges) and maps of sets $s_G, t_G : E_G \rightarrow V_G$ source and target maps of directed edges
- natural transformation $\eta : G \rightarrow G'$ is usual notion of map of directed graphs

$$\eta_V : V_G \rightarrow V_{G'} \quad \text{and} \quad \eta_E : E_G \rightarrow E_{G'}$$

preserving incidence relations

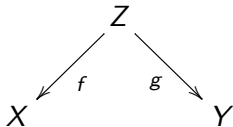
$$\eta_V \circ s_G = s_{G'} \circ \eta_E \quad \text{and} \quad \eta_V \circ t_G = t_{G'} \circ \eta_E$$

some categorical notions for later use

- **monomorphism**: morphism $f : X \rightarrow Y$ such that for every pair $g_1, g_2 : Z \rightarrow X$

$$(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$$

- **span** (also called **correspondence**) between objects X, Y is a pair of morphisms



- **jointly monic span** (also called **internal relation**): span such that for any pair $h_1, h_2 : W \rightarrow Z$ if $f \circ h_1 = f \circ h_2$ and $g \circ h_1 = g \circ h_2$ then $h_1 = h_2$

how to compare categories

- there is a notion of “isomorphism” $\mathcal{C} \cong \mathcal{C}'$: two functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$ that are inverses of each other
- **but**: this notion is *too strong*, does not correctly reflect the idea of two categories being “essentially the same”
- *better notion* **equivalence of categories**: two functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$ and two invertible natural transformations $\xi : F \circ G \rightarrow \text{id}_{\mathcal{C}}$ and $\eta : G \circ F \rightarrow \text{id}_{\mathcal{C}'}$
- category \mathcal{C} and $\text{Sk}(\mathcal{C})$ skeleton (subcategory) with objects a representative for each isomorphism classes of objects of \mathcal{C} (Axiom of Choice), inclusion $\text{Sk}(\mathcal{C}) \subset \mathcal{C}$ **full** subcategory ($\text{Hom}_{\text{Sk}(\mathcal{C})}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$) and **essentially surjective** (each object in \mathcal{C} is isomorphic to an object in $\text{Sk}(\mathcal{C})$)
- \mathcal{C} is equivalent to its skeleton $\text{Sk}(\mathcal{C})$

Elementary Theory of the Category of Sets (ETCS)

- How to use categories for axiomatic set theory without ending up with some circular definition?
- Lawvere's ETCS is a modern alternative to ZFC
- **first step**: we want a **Theory of Categories** $T(\text{Cat})$
- **language** $\mathcal{L}_{\text{Cat}} = \{\Phi, \mathcal{R}\}$ with **unary function symbols** $s, t \in \Phi$ and a **ternary relation** $c \in \mathcal{R}$
- **terms** f, g, \dots (morphisms) with $s(f), t(f)$ here meaning the identity morphisms $1_{s(f)}, 1_{t(f)}$ that corresponds to the objects $s(f), t(f)$
- relation models **composition**

$$c(f, g, h) \quad \text{stands for} \quad h = f \circ g$$

- Note: instead of objects and morphisms, just use **morphisms** (keep track of objects through their identity morphisms):
object term: term e such that $e = s(e)$, $e = t(e)$

- **axioms** of $T(\text{Cat})$:

1. $\vdash (ss(f) = s(f) = ts(f)) \wedge (tt(f) = t(f) = st(f)).$

2. $c(f, g, h) \vdash (s(f) = t(g) \wedge t(f) = t(h) \wedge s(g) = s(h))$

3. $s(f) = t(g) \vdash \exists!_h c(f, g, h)$

4. (**Identity** laws) $\vdash c(f, s(f), f) \wedge c(t(f), f, f)$

5. (**Associativity**) $(c(f, g, j) \wedge c(g, h, k)) \vdash (c(j, h, m) \Leftrightarrow c(f, k, m))$

- **second step**: refining the theory $T(\text{Cat})$ to a theory of **finitely complete categories** $T(\text{Lex})$ (over same language \mathcal{L}_{Cat})

- **axioms** of $T(\text{Lex})$: same as axioms of $T(\text{Cat})$ with two additional axioms

- 1 existence of terminal object
- 2 existence of pullbacks

$$1. \exists_1(1 = s(1)) \wedge (\forall_f(f = s(f)) \Rightarrow (\exists!g: (s(g) = f \wedge t(g) = 1)))$$

2. (Pullbacks exist) Define $p(f, g, h, k)$ to be the conjunction of the formulas

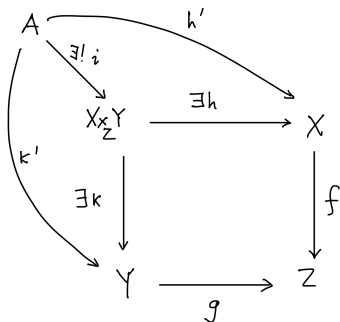
$$(\exists_j c(f, h, j) \wedge c(g, k, j))$$

$$(\exists_{j'} (c(f, h', j') \wedge c(g, k', j'))) \Rightarrow (\exists!i: (c(h, i, h') \wedge c(k, i, k'))))$$

Then

$$t(f) = t(g) \vdash \exists_h \exists_k p(f, g, h, k)$$

existence of pullback: universal property



given morphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ there is an object $X \times_Z Y$ (fibered product) with morphisms $h : X \times_Z Y \rightarrow X$ and $k : X \times_Z Y \rightarrow Y$, with $f \circ h = g \circ k$ such that for any object A and morphisms $h' : A \rightarrow X$ and $k' : A \rightarrow Y$ with $f \circ h' = g \circ k'$ there is a *unique* $i : A \rightarrow X \times_Z Y$ that makes the diagram commutative

optimal with respect to these diagrams

- **third step**: extend to a **Theory of Elementary Topoi** $T(\text{Top})$
- extend **language** $\mathcal{L}_{\text{Cat}} = \{s, t; c\}$ with three additional unary symbols

$$\mathcal{L}_{\text{Top}} = \{s, t, P, \lambda, \rho; c\}$$

- symbol P used to provide a notion of **power set**
- symbols ρ and λ used to build a **relation** (between a set and its power set) replacing the notion of “membership” for sets
- formula $M(f, g)$ in the language \mathcal{L}_{Cat} that expresses the property that f, g are a **jointly monic span**

$$s(f) = s(g) \wedge (\forall_{h,i,j,k}(c(f, h, j) \wedge c(f, i, j) \wedge c(g, h, k) \wedge c(g, i, k)) \Rightarrow h = i)$$

- **axioms** of the theory $T(\text{Top})$ of elementary topoi

- 1 there is $P(f : X \rightarrow Y) = P(Y)$ an object (depending only on $t(f)$) and a span $(\lambda(f), \rho(f))$ between $P(Y)$ and Y :

$$\vdash (P(f) = P(t(f)) = t(P(f)) = t(\lambda(f))) \wedge (t(\rho(f)) = t(f))$$

- 2 $M(\lambda(f), \rho(f))$ (i.e. they form a jointly monic span)
- 3 **universal property of power objects** $P(X)$

$$M(f, g) \vdash \exists! h \exists k \rho(h, \lambda(f), g, k) \wedge c(\rho(f), k, f)$$

What are this universal property and these axioms saying?

they are supposed to capture the “set-like” nature of the objects of this category, how so?

unraveling the topos axioms

- the concept of “being a set” here boils down to the possibility of forming an associated **power set**
- all other axioms and properties for $T(\text{Cat})$ and $T(\text{Lex})$ are shared by many other kinds of objects that are not sets
- what is “set-like” is just these additional axioms of $T(\text{Top})$

key observation: *membership as correspondence*

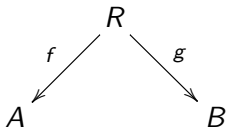
- in the usual way of thinking about a set, **membership** \in can be seen as a **correspondence** between a set X and its power set $\mathcal{P}(X)$

$$\begin{array}{ccc} \mathcal{E}(X) = \{(a, S) \mid a \in S\} \subset X \times \mathcal{P}(X) & & \\ \swarrow \pi_X & & \searrow \pi_{\mathcal{P}(X)} \\ a \in X & & S \in \mathcal{P}(X) \end{array}$$

- what are the **properties that characterize** this membership-correspondence?

$$\mathcal{E}(X) = \{(a, S) \mid a \in S\} \subset X \times \mathcal{P}(X)$$

- consider **any correspondence** $R \subseteq A \times B$ (or more generally $R \rightarrow A \times B$)



- R determines a **characteristic map** $\chi_R : B \rightarrow \mathcal{P}(A)$

$$\chi_R : b \mapsto \{a \in A \mid (a, b) \in R\}$$

in case of $R \subseteq A \times B$

- for more general case $(f, g) : R \rightarrow A \times B$

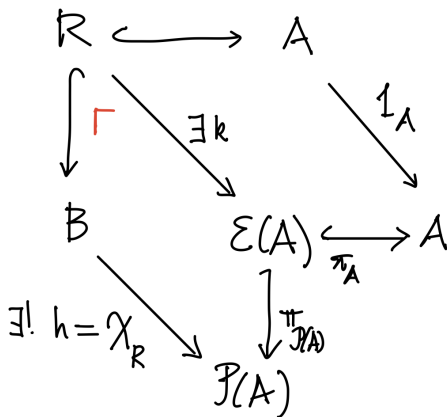
$$\chi_R(b) = \{a \in A \mid (a, b) = (f(r), g(r)) \text{ for some } r \in R\}$$

universal property of power set:

there is a bijection between correspondences $R \subseteq A \times B$ and maps $B \rightarrow \mathcal{P}(A)$

- $R \mapsto \chi_R$ works for arbitrary span R , but reverse direction $\chi : B \rightarrow \mathcal{P}(A)$ to R with $\chi = \chi_R$ requires more restrictive correspondences (because χ_R only depend on image $(f, g)(R) \subset A \times B$ not on R itself).
- so need some **injectivity assumption** for the span defining R to reconstruct R from the map $\chi : B \rightarrow \mathcal{P}(A)$ as $R_\chi = \{(a, b) \in A \times B \mid a \in \chi(b)\}$

correspondence $R \subseteq A \times B$ is **pullback** of the membership correspondence $\mathcal{E}(A)$ along $1_A \times \chi_R$ with $\chi_R : B \rightarrow \mathcal{P}(A)$ the characteristic map $\chi_R : B \rightarrow \mathcal{P}(A)$ representing R



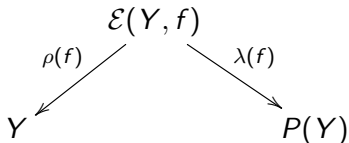
“ \square ” notation for square that is a pullback diagram

$$R = B \times_{\mathcal{P}(A)} \mathcal{E}(A) = \{(b, (a, S)) \mid \chi_R(b) = S\} = \{(a, b) \mid a \in \chi_R(b)\}$$

$$k : (a, b) \in R \mapsto (a, \chi_R(b)) \in \mathcal{E}(A)$$

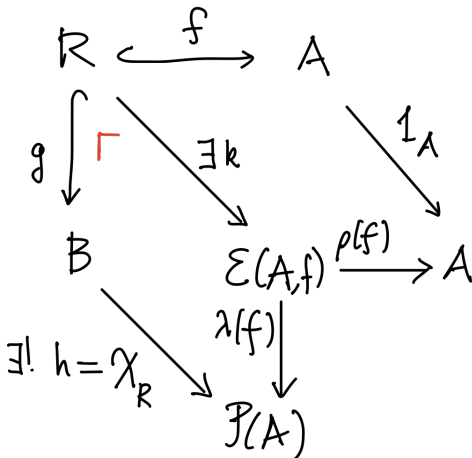
meaning of topos axioms

- 1 **power object** $P(f : X \rightarrow Y) = P(Y)$ comes with a “membership correspondence”



- 2 **jointly monic span** means can think of $\mathcal{E}(Y, f)$ as $(\rho(f), \lambda(f)) : \mathcal{E}(Y, f) \hookrightarrow Y \times P(Y)$

- 3 **universal property of power object:** given another $M(f, g)$ meaning any correspondence $(f, g) : R \hookrightarrow A \times B$ there is a unique $h = \chi_R : B \rightarrow P(A)$ such that R is obtained as pullback of the membership correspondence $\mathcal{E}(A, f)$ by diagram



Lawvere's ETCS

- define **theory** $T(\text{ETCS})$
- **language** $\mathcal{L}_{\text{ETCS}}$ extend \mathcal{L}_{Top} with **constant symbols**: $\mathbb{N}, 0, \sigma$
- **axioms**: same axioms as $T(\text{Top})$ with additional:

1. (Well-pointedness: equality of maps $x \rightarrow y$ is detected by points $1 \rightarrow x$)

$$(s(f) = s(g) \wedge t(f) = t(g)) \vdash \forall_h (s(h) = 1 \wedge t(h) = s(f) \Rightarrow f \circ h = g \circ h) \Rightarrow f = g$$

2. (Axiom of Choice: every epi f admits a section i)

$$(\forall_{g,h} (t(f) = s(g) = s(h) \wedge t(g) = t(h) \wedge g \circ f = h \circ f) \Rightarrow g = h) \vdash \exists_i (s(i) = t(f) \wedge t(i) = s(f) \wedge f \circ i = t(f))$$

3. (Zero and successor)

$$\vdash s(0) = 1 \wedge t(0) = \mathbb{N} = s(\sigma) = t(\sigma)$$

4. (Recursion: existence)

$$s(f) = 1 \wedge t(f) = s(g) = t(g) \vdash \exists_h (s(h) = \mathbb{N} \wedge t(h) = t(f) \wedge f = h \circ 0 \wedge h \circ \sigma = g \circ h)$$

5. (Recursion: uniqueness)

$$(s(f) = \mathbb{N} = s(g) \wedge t(f) = t(g) \wedge f \circ 0 = g \circ 0 \wedge f \circ \sigma = g \circ \sigma) \vdash f = g$$

comments on ETCS axioms

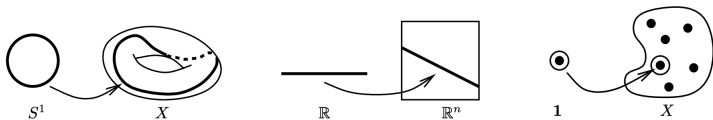
- **NNS** (natural number system): zero and successor and recursion

whenever X is a set, $a \in X$, and $r: X \rightarrow X$,
there is a unique function $x: N \rightarrow X$ such that
 $x(0) = a$ and $x(s(n)) = r(x(n))$ for all $n \in N$.

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{0} & N & \xrightarrow{s} & N \\ \downarrow 1_1 & & \downarrow x & & \downarrow x \\ \mathbf{1} & \xrightarrow{a} & X & \xrightarrow{r} & X \end{array}$$

this expresses the general property that given $a \in X$ and
 $f: X \rightarrow X$ one can form the sequence $\{f^n(a)\}$ of iterates

- **well-pointedness:** “points” versus “generalized points”



generalized points pick out certain kinds of “shapes” in a given X

- in general (depending on what kinds of objects one is dealing with) one might expect that shapes more general than points may be needed to determine X completely (e.g. loops that carry global information about the topology of X)
- “points” in the strict sense $x : \{\star\} \rightarrow X$ carry only very local information about X so well-pointedness is the statement that the kind of objects being considered only require this kind of local information to be determined
- it’s OK if we want objects with no structure (sets) not OK if objects have structure involving more global conditions

Remark on ZFC vs ETCS: ETCS does not have an analog of the “Axiom Scheme of Replacement” of ZFC

role of the Axiom of Choice

- every surjection $f : X \rightarrow Y$ has a section $s : Y \rightarrow X$ with $f \circ s = 1_Y$

what role does it play?

- a topos determines a logic
- this logic is intuitionistic
- assuming the axiom of choice implies the excluded middle
 - Radu Diaconescu, *Axiom of Choice and complementation*, Proceedings AMS, 51 (1975) N.1, 176–178
- so it forces the logic of the topos to be classical

Topoi and Subobject classifier

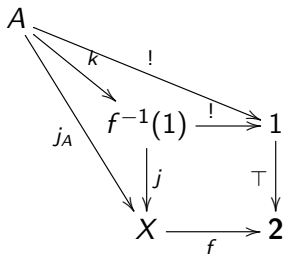
for Sets

- subsets $A \subseteq X$ correspond to functions $\chi_A : X \rightarrow \{0, 1\}$ (truth values) detecting if $x \in X$ belongs to A or not (characteristic function of A)
- call $\mathbf{2} = \{0, 1\}$ the truth values object
- $\chi_A : X \rightarrow \mathbf{2}$ fits in commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{!} & \mathbf{1} \\ \downarrow j_A & & \top \downarrow \\ X & \xrightarrow{\chi_A} & \mathbf{2} \end{array}$$

with $! : A \rightarrow \mathbf{1}$ unique morphism to terminal object and $\top : \mathbf{1} \rightarrow \mathbf{2}$ the true value and $j_A : A \hookrightarrow X$

- the diagram is a **pullback**: consider diagram



square diagram is the pullback; commutativity of outer diagram gives that $x \in A$ has $f(x) = 1$ so $A \subseteq f^{-1}(1)$, and if $f = \chi_A$ then exactly $A = f^{-1}(1)$ and map $k = 1_A$

- characteristic functions χ_A give isomorphism $\mathcal{P}(X) \simeq \mathbf{2}^X = \text{Hom}(X, \mathbf{2})$

$$A \in \mathcal{P}(X) \mapsto \chi_A \in \mathbf{2}^X$$

- so $\mathbf{2}$ is a **subobject classifier**: the datum $\mathbf{2}$ together with $\top : 1 \rightarrow \mathbf{2}$ determine the assignment of χ_A to a subobject $A \subseteq X$ and this realizes $\mathcal{P}(X)$ (that classifies subobjects of X) as the set $\text{Hom}(X, \mathbf{2})$.
- all **power sets** are realized as sets of morphisms to a **fixed** truth-value object $\mathbf{2}$

for a more general topos \mathcal{T}

- generalize this to a topos \mathcal{T} (that need not be Sets)
- need an **object** Ω of \mathcal{T} that plays the role of the set $\{0, 1\}$ of **truth values**
- need a **point** $\top : 1 \rightarrow \Omega$ (with 1 the terminal object of \mathcal{T}), a point of Ω that represent “**true**”
- Ω should have the property that every **monomorphism** $j_A : A \hookrightarrow X$ in \mathcal{T} (think of these as “**subobjects**” of X) fits in a **pullback** diagram

$$\begin{array}{ccc} A & \xrightarrow{!} & 1 \\ j_A \downarrow & & \downarrow \top \\ X & \xrightarrow{\chi_{j_A}} & \Omega \end{array}$$

the monomorphism j_A is the pullback of $\top : 1 \rightarrow \Omega$ along the characteristic morphism χ_{j_A}

- such object Ω (together with $\top : 1 \rightarrow \Omega$) is called a **subobject classifier**

- suppose in \mathcal{T} there is an object A of sentences of the language \mathcal{L}_{Top} (meaning there is an identification of sentences with morphisms $\alpha : 1 \rightarrow A$)
- then a truth valuation is a morphism $\nu : A \rightarrow \Omega$ and ν **satisfies** a sentence α

$$\nu \models \alpha$$

if (in Sets: $\nu(\alpha) = \top(1)$) have diagram

$$\begin{array}{ccc}
 & 1 & \\
 \alpha \swarrow & & \searrow \top \\
 A & \xrightarrow{\nu} & \Omega
 \end{array}$$

- α **tautology** if satisfies by all valuations $\nu : A \rightarrow \Omega$
- Ω has operations $\wedge, \vee, \neg, \Rightarrow$ induced through ν from those in \mathcal{L}_{Top} (truth tables: effect of logical operations on truth values)

$$\neg \nu(\alpha) := \nu(\neg \alpha), \quad \nu(\alpha) \wedge \nu(\beta) := \nu(\alpha \wedge \beta), \quad \text{etc}$$

- what happens when the subobject classifier Ω is not $\mathbf{2} = \{0, 1\}$ (a different kind of truth values): what kind of logic does this correspond to?

the subobject classifier is a **Heyting algebra object**

what this gives for Sets with $\Omega = \mathbf{2}$

- $\neg : \mathbf{2} \rightarrow \mathbf{2}$ is characteristic function χ_A of subset $A = \{0\} \subset \{0, 1\}$ because sends 0 to 1 and 1 to 0; can also identify $\{0\} \hookrightarrow \{0, 1\}$ with morphism $\perp : 1 \rightarrow \{0, 1\}$ sending 1 to the point 0
- truth table for \wedge shows it is characteristic function χ_A of subset $A = \{(1, 1)\} \subset \mathbf{2} \times \mathbf{2}$

$$\wedge(0, 0) = 0,$$

$$\wedge(0, 1) = 0,$$

$$\wedge(1, 0) = 0,$$

$$\wedge(1, 1) = 1,$$

morphism $(\top, \top) : 1 \rightarrow \mathbf{2} \times \mathbf{2}$ mapping 1 to element $(1, 1)$

- case of \vee : truth table identifies with characteristic function of the subset $U = \{(0, 1), (1, 0), (1, 1)\} \subset \mathbf{2} \times \mathbf{2}$; expressible in categorical terms as the pullback

$$\begin{array}{ccc}
 U & \xrightarrow{!} & \mathbf{1} \\
 \downarrow j_U & & \top \downarrow \\
 \mathbf{2} \times \mathbf{2} & \xrightarrow{\vee} & \mathbf{2}
 \end{array}$$

- order \leq has same truth table as \Rightarrow

$$\leq, \Rightarrow: (0, 0) \mapsto 1, (0, 1) \mapsto 1, (1, 0) \mapsto 0, (1, 1) \mapsto 1$$

pullback diagram: characteristic function of set $\{\leq\} := \{(0, 0), (0, 1), (1, 1)\}$

$$\begin{array}{ccc}
 \{\leq\} & \xrightarrow{!} & \mathbf{1} \\
 \downarrow j & & \top \downarrow \\
 \mathbf{2} \times \mathbf{2} & \xrightarrow{\Rightarrow} & \mathbf{2}
 \end{array}$$

- $\{\leq\}$ also described as an **equalizer**

- **equalizer**: given two morphisms $f, g : A \rightrightarrows B$, object E with monomorphism $i : E \rightarrow A$ such that $f \circ i = g \circ i$ and *optimal*: for any other $h : C \rightarrow A$ with $f \circ h = g \circ h$ exists unique $k : C \rightarrow E$ with $i \circ k = h$

$$\begin{array}{ccccc}
 E & \xrightarrow{i} & A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\
 \uparrow \exists! k & & \nearrow h & & \\
 C & & & &
 \end{array}$$

- set $\{\leq\} = \{(0,0), (0,1), (1,1)\}$ is equalizer of morphisms $\wedge : \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ and $\pi_1 : \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ (projection on first factor), because it's the subset on which these agree: on complement $(1,0)$ have $1 \wedge 0 = 0 \neq \pi_1(1,0) = 1$
- in general **define partial order \leq as equalizer** of \wedge and π_1

$$\leq \xrightarrow{i} \Omega \times \Omega \begin{array}{c} \xrightarrow{\wedge} \\ \xrightarrow{\pi_1} \end{array} \Omega$$

useful facts about topoi

Epi-mono decomposition

- In any topos \mathcal{T} every morphism $f : X \rightarrow Y$ factors uniquely (up to isomorphism) $f = m \circ e$ as an epimorphism $e : X \twoheadrightarrow Z$ followed by a monomorphism $m : Z \hookrightarrow Y$

other properties: (follow from the topos axioms)

- a topos \mathcal{T} has all finite limits (from having pullbacks and a terminal object)
- hence it has equalizers
- \mathcal{T} also has all finite colimits (more subtle: using finite limits and power-objects)
- hence \mathcal{T} also has an **initial object** 0 (in the case of Sets $0 = \emptyset$) and has **coproduct** (disjoint union for Sets) and **coequalizers**

coequalizer

- morphisms $f, g : X \rightarrow Y$, object Q and epimorphism $q : Y \rightarrow Q$ with $q \circ f = q \circ g$, *optimal*: for any other $q' : Y \rightarrow Q'$ with $q' \circ f = q' \circ g$ there is a unique $u : Q \rightarrow Q'$ with commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{q} & Q \\ & \xrightarrow{g} & & & \downarrow \text{---} u \\ & & & \searrow q' & Q' \end{array}$$

topoi are extensive categories

- coproducts “interact well” with pullbacks
- meaning, in a diagram

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A + B & \longleftarrow & B \end{array}$$

the two squares are pullbacks iff the top row is a coproduct

revisit case of $\vee : \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2}$ in Sets:

- because have **coproduct** \sqcup have commutative diagram

$$\begin{array}{ccccc} 1 \times \mathbf{2} \simeq \mathbf{2} & \longrightarrow & \mathbf{2} \sqcup \mathbf{2} & \longleftarrow & \mathbf{2} \simeq \mathbf{2} \times 1 \\ & \searrow & \downarrow & \swarrow & \\ & (\top, 1_2) & & (1_2, \top) & \\ & & \mathbf{2} \times \mathbf{2} & & \end{array}$$

with induced map $((\top, 1_2) \sqcup (1_2, \top)) : \mathbf{2} \sqcup \mathbf{2} \rightarrow \mathbf{2} \times \mathbf{2}$

- this map has **epi-mono** factorization

$$\begin{array}{ccc} \mathbf{2} \sqcup \mathbf{2} & \xrightarrow{((\top, 1_2) \sqcup (1_2, \top))} & \mathbf{2} \times \mathbf{2} \\ & \searrow e & \swarrow m \\ & U & \end{array}$$

in Sets e is map to image and m is inclusion of image, and image U here is set with $\vee = \chi_U$

$$U = \{(0, 1), (1, 0), (1, 1)\}$$

also revisit $\neg : \mathbf{2} \rightarrow \mathbf{2}$ in Sets:

- as seen, there is a false-value morphism $\perp : \mathbf{1} \rightarrow \mathbf{2}$ with (for $\Omega = \mathbf{2}$)

$$\begin{array}{ccc}
 \mathbf{1} & \xrightarrow{\quad ! \quad} & \mathbf{1} \\
 \downarrow \perp & & \downarrow \top \\
 \mathbf{2} & \xrightarrow{\quad \neg \quad} & \mathbf{2}
 \end{array}$$

- this can also be characterized as the arrow that makes this a pullback diagram

$$\begin{array}{ccc}
 \mathbf{0} & \xrightarrow{\quad ! \quad} & \mathbf{1} \\
 \downarrow & & \downarrow \top \\
 \mathbf{1} & \xrightarrow{\quad \perp \quad} & \Omega
 \end{array}$$

i.e. in $\Omega = \mathbf{2}$ image of \perp is **complement** of image of \top

- in case of Sets, the subobject classifier $\mathbf{2}$ is in fact a **Boolean algebra** with these $(\leq, \wedge, \vee, \neg, \perp, \top)$ (just the basic Boolean semiring $\mathbb{B} = \{0, 1\}$ interpreted in this categorical way)

what happens in a general topos \mathcal{T} ? $(\mathcal{T}, \leq, \wedge, \vee, \perp, \top)$ **Heyting**

- now Ω is not a set, just an object of \mathcal{T}
- $\wedge : \Omega \times \Omega \rightarrow \Omega$ is the morphism that makes this diagram a pullback

$$\begin{array}{ccc}
 \mathbf{1} & \overset{!}{\dashrightarrow} & \mathbf{1} \\
 \langle \top, \top \rangle \downarrow & & \downarrow \top \\
 \Omega \times \Omega & \xrightarrow{\wedge} & \Omega
 \end{array}$$

- \leq is the **equalizer** of \wedge and first projection $\pi_1 : \Omega \times \Omega \rightarrow \Omega$

$$\leq \overset{i}{\hookrightarrow} \Omega \times \Omega \overset{\wedge}{\underset{\pi_1}{\rightrightarrows}} \Omega$$

- $\vee : \Omega \times \Omega \rightarrow \Omega$ is the morphism that makes this diagram a pullback

$$\begin{array}{ccc}
 U & \overset{!}{\dashrightarrow} & \mathbf{1} \\
 \downarrow \Upsilon & & \downarrow \top \\
 \Omega \times \Omega & \xrightarrow{\vee} & \Omega,
 \end{array}$$

where the object U is the epi-mono factorization of the map from coproduct to product

$$\begin{array}{ccc}
 \Omega + \Omega & \overset{\langle \langle \top, \text{id} \rangle, \langle \text{id}, \top \rangle \rangle}{\dashrightarrow} & \Omega \times \Omega \\
 & \searrow & \nearrow \\
 & U &
 \end{array}$$

- the 0 and 1 elements of the Heyting algebra are the morphisms $\perp: 1 \rightarrow \Omega$ and $\top: 1 \rightarrow \Omega$ where \top is the subobject-classifier morphism (true-value) and \perp is the **characteristic function of the initial object**
- by sub-object classifier universal property: for any monomorphism $m: A \hookrightarrow X$ in \mathcal{T} there is a **characteristic morphism** χ_m such that **pullback**:

$$\begin{array}{ccc}
 A & \xrightarrow{!} & 1 \\
 m \downarrow & & \downarrow \top \\
 X & \xrightarrow{\chi_m} & \Omega
 \end{array}$$

- in particular apply to unique morphism from initial to terminal $0 \rightarrow 1$ to define $\perp: 1 \rightarrow \Omega$ as characteristic morphism

$$\begin{array}{ccc}
 A & \xrightarrow{!} & 1 \\
 ! \downarrow & & \downarrow \top \\
 1 & \xrightarrow{\perp := \chi!} & \Omega
 \end{array}$$

- then the intuitionistic implication $\xrightarrow[\mathcal{I}]{!} : \Omega \times \Omega \rightarrow \Omega$ of this Heyting algebra given by the characteristic morphism of the \leq equalizer

$$\begin{array}{ccc}
 \leq & \xrightarrow{!} & \mathbf{1} \\
 \downarrow \Upsilon & & \downarrow \top \\
 \Omega \times \Omega & \xrightarrow{\Rightarrow} & \Omega,
 \end{array}$$

- the intuitionistic negation $\xrightarrow[\mathcal{I}]{\neg} : \Omega \rightarrow \Omega$ of this Heyting algebra given by the characteristic morphism

$$\begin{array}{ccc}
 \mathbf{1} & \xrightarrow{!} & \mathbf{1} \\
 \downarrow \Upsilon & & \downarrow \top \\
 \perp & & \\
 \downarrow & & \\
 \Omega & \xrightarrow{\neg} & \Omega,
 \end{array}$$

- the topos \mathcal{T} is **Boolean** if $\overline{\quad} \circ \overline{\quad} = 1_{\Omega}$, so Ω is a **Boolean algebra**
- in general Ω is a **Heyting algebra** not Boolean, so in general a topos logic is intuitionistic not classical

in ETCS: also assume **Axiom of Choice**

- AoC: **every epimorphism has a section**
- **complemented subobject**: $m : A \hookrightarrow X$ monomorphism in \mathcal{T} such that $\exists A^c$ object and monomorphism $n : A^c \hookrightarrow X$ disjoint from m , namely pullback:

$$\begin{array}{ccc}
 0 & \xrightarrow{!} & A^c \\
 \downarrow ! & & \downarrow n \\
 A & \xrightarrow{m} & X
 \end{array}$$

- AoC implies all subobjects are complemented (Diaconescu)
- take a monomorphism $m : A \hookrightarrow X$
- $i_1, i_2 : X \rightarrow X + X$ maps to coproduct, and **coequalizer**

$$A \begin{array}{c} \xrightarrow{i_1 \circ m} \\ \rightrightarrows \\ \xrightarrow{i_2 \circ m} \end{array} (X + X) \xrightarrow{q} Q =: X +_A X$$

(in Sets: two copies of X glued together along A)

- by AoC epimorphism $p : X + X \rightarrow Q$ has a section $s : Q \rightarrow X + X$ with $p \circ s = 1_Q$
- **topoi are extensive categories** implies *subobjects of coproducts are coproducts*
- so use section s to decompose $Q = Q_1 + Q_2$ with $Q_1 = s^{-1}(X + 0)$ and $Q_2 = s^{-1}(0 + X)$

- then take commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{m} & X \\
 m \downarrow & & \downarrow j_1 \\
 X & \xrightarrow{j_2} & X +_A X
 \end{array}$$

- take compositions of section s with $j_1 = p \circ i_1$ and $j_2 = p \circ i_2$ monomorphisms

$$X \begin{array}{c} \xrightarrow{s \circ j_1} \\ \rightrightarrows \\ \xrightarrow{s \circ j_2} \end{array} X +_A X = Q_1 + Q_2$$

- same reason (extensive category) X subobject of $Q_1 + Q_2$ splits (in two ways)

$$X = B_1 + C_1 \quad \text{with } B_1 = j_1^{-1}(Q_1), \quad C_1 = j_1^{-1}(Q_2)$$

$$X = B_2 + C_2 \quad \text{with } B_2 = j_2^{-1}(Q_1), \quad C_2 = j_2^{-1}(Q_2)$$

- then get decomposition $A = A_1 + A_2$

$$\begin{array}{ccc}
 A & \xrightarrow{m} & B_1 + C_1 \\
 m \downarrow & & \downarrow j_1 \\
 B_2 + C_2 & \xrightarrow{j_2} & Q_1 + Q_2
 \end{array}$$

$$\begin{array}{ccc}
 A_1 & \longrightarrow & B_1 \\
 \downarrow & & \downarrow j_1 \\
 B_2 & \xrightarrow{j_2} & Q_1
 \end{array}
 \qquad
 \begin{array}{ccc}
 A_2 & \longrightarrow & C_1 \\
 \downarrow & & \downarrow j_1 \\
 C_2 & \xrightarrow{j_2} & Q_2
 \end{array}$$

- in Sets $A_1 = B_1 \cap B_2$ and $A_2 = C_1 \cap C_2$ with $A = A_1 + A_2$
- **complement** $A^c = B_1 \cap C_2 + B_2 \cap C_1$

What are examples of topoi \mathcal{T} that are not Sets?

- presheaf category of a small category \mathcal{C} : category of functors $\text{Fun}(\mathcal{C}^{op}, \text{Sets})$ (used, in many settings, including as a model of quantum theory)
- topology: sheaves of sets on a topological space (with $\Omega(U) = \{V \text{ open set} \mid V \subseteq U\}$)
- algebraic geometry: étale topos of a scheme (also a category of sheaves)
- topos of trees (programming language semantics)

all these behave “like sets” and have their (intuitionistic) logics

presheaves on a topological space

Definition

Let X be a topological space. A **presheaf** \mathcal{F} on X consists of the data:

- (i) for every open subset U of X , a set $\mathcal{F}(U)$ and
- (ii) for every inclusion $V \subseteq U$ of open subsets of X , a function $\rho_{U,V} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ subject to the conditions
 - $\rho_{U,U}$ is the identity map $\mathcal{F}(U) \rightarrow \mathcal{F}(U)$ and
 - if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The maps $\rho_{U,V}$ are called **restriction maps**, and we sometimes write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.

A **morphism of presheaves** $\mathcal{F} \rightarrow \mathcal{G}$ on a topological space X is a collection of maps $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ which is compatible with respect to restriction maps.

idea: think of $\mathcal{F}(U)$ as a “set of functions” on U and $\rho_{U,V}$ as “restriction of functions” from U to $V \subseteq U$

Definition

A **sheaf** \mathcal{F} on a topological space X is a presheaf on X satisfying the additional conditions

- (i) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U , and if $s, t \in \mathcal{F}(U)$ are elements such that $s|_{V_i} = t|_{V_i}$ for all i , then $s = t$;
- (ii) if U is an open set, if $\{V_i \mid i \in I\}$ is an open covering of U , and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i , with the property that for each $i, j \in I$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ (necessarily unique by (i)) such that $s|_{V_i} = s_i$ for each i .

A morphism of sheaves is defined as a morphism of the underlying presheaves.

- **idea**: think of presheaves as “local data” assigned at open sets, with no consistency over “patching together” overlapping open sets, and sheaves as those local data that patch consistency and can define “global data” on the underlying X

versions for categories

- \mathcal{C} a **small category**
- **presheaf** on \mathcal{C} is a function $P : \mathcal{C}^{op} \rightarrow \text{Sets}$
- $\text{Fun}(\mathcal{C}^{op}, \text{Set})$ also written as $[\mathcal{C}^{op}, \text{Sets}]$ is the category of presheaves on \mathcal{C}
- morphisms are natural transformations of functors

idea: a presheaf over \mathcal{C} is a family of sets parameterized by the objects of \mathcal{C} that change according to the morphisms in \mathcal{C} , so one can think of it as “a set that evolves according to \mathcal{C} ”

in order to pass to sheaves, on a category, need a notion analogous to open coverings of a topological space, and matching conditions for presheaves data: this is done through the notion of **Grothendieck topology** on a category

sieves: an abstract notion of open covering in categories

Definition

- Given a category \mathcal{C} and an object $c \in \text{Ob}(\mathcal{C})$, a **presieve** P in \mathcal{C} on c is a collection of arrows in \mathcal{C} with codomain c .
- Given a category \mathcal{C} and an object $c \in \text{Ob}(\mathcal{C})$, a **sieve** S in \mathcal{C} on c is a collection of arrows in \mathcal{C} with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense.

- We say that a sieve S is **generated** by a presieve P on an object c if it is the smallest sieve containing it, that is if it is the collection of arrows to c which factor through an arrow in P .

If S is a sieve on c and $h: d \rightarrow c$ is any arrow to c , then

$$h^*(S) := \{g \mid \text{cod}(g) = d, h \circ g \in S\}$$

is a sieve on d .

- topos $[\mathcal{C}^{op}, \text{Sets}]$ of presheaves on a small category \mathcal{C}
- terminal object 1 the constant functor that maps every object $c \in \mathcal{C}$ to the set $\{\star\}$ and every morphism to $1_{\{\star\}}$
- sieves in \mathcal{C} on c form a poset with minimal (empty) and maximal (all morphisms with target c) sieves
- the **subobject classifier** Ω is the presheaf $\Omega(c) = \text{Sieves}(c)$ the set of sieves on $c \in \text{Obj}(\mathcal{C})$
- the **true value** $\top : 1 \rightarrow \Omega$ has $1_c : \{\star\} \rightarrow \text{Sieves}(c)$ that selects the maximal sieve (of all morphisms to c)
- now there are intermediate truth values between false (the empty sieve) and true (the maximal sieve)

- **topos** $\text{Sh}(X)$ of sheaves on a topological space X
- terminal object 1 is the sheaf with $1(U) = \{\star\}$ for each open set U
- the **subobject classifier** Ω is given by the sheaf that assigns to an open set $U \subseteq \Omega$ the set

$$\Omega(U) = \{V \subseteq U \mid V \text{ open}\}$$

of all open subsets of U

- truth morphism $\top : 1 \rightarrow \Omega$ has $\top_U : 1(U) \rightarrow \Omega(U)$ given by $\top_U(\star) = U$
- interpretation as **truth values**: the truth value assigns open set over which true
- the characteristic function $\chi_m : F \rightarrow \Omega$ of a monomorphism $m : G \rightarrow F$ is given by $\chi_{m,U} : F(U) \rightarrow \Omega(U)$ assigning to $x \in F(U)$ the open set $\chi_{m,U}(x)$ given by the union of all the open sets $V \subseteq U$ such that the restriction $x|_V \in G(V)$

for more general categories **sieves and Grothendieck topologies**

- sieves (which generalize the idea of an open covering) are used to define topologies on categories (Grothendieck topologies)

Definition

- A **Grothendieck topology** on a category \mathcal{C} is a function J which assigns to each object c of \mathcal{C} a collection $J(c)$ of sieves on c in such a way that
 - (i) (**maximality axiom**) the maximal sieve $M_c = \{f \mid \text{cod}(f) = c\}$ is in $J(c)$;
 - (ii) (**stability axiom**) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \rightarrow c$;
 - (iii) (**transitivity axiom**) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \rightarrow c$ in S , then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of \mathcal{C} are said to be **J -covering**.

- A **site** is a pair (\mathcal{C}, J) where \mathcal{C} is a small category and J is a Grothendieck topology on \mathcal{C} .

Definition

- A **presheaf** on a (small) category \mathcal{C} is a functor $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf on \mathcal{C} and S be a sieve on an object c of \mathcal{C} .

A **matching family** for S of elements of P is a function which assigns to each arrow $f : d \rightarrow c$ in S an element $x_f \in P(d)$ in such a way that

$$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \rightarrow d .$$

An **amalgamation** for such a family is a single element $x \in P(c)$ such that

$$P(f)(x) = x_f \quad \text{for all } f \text{ in } S .$$

not all presheaves are sheaves: there can be matching families (compatible local determinations) that do not extend by amalgamation to a single global object

Grothendieck topoi, categories of sheaves on a site

- Given a site (\mathcal{C}, J) , a presheaf on \mathcal{C} is a **J -sheaf** if every matching family for any J -covering sieve on any object of \mathcal{C} has a unique amalgamation.
- The category $\mathbf{Sh}(\mathcal{C}, J)$ of **sheaves on the site** (\mathcal{C}, J) is the full subcategory of $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ on the presheaves which are J -sheaves.
- A **Grothendieck topos** is **any category equivalent to the category of sheaves on a site**.

Credits

These slides incorporate material taken from:

- Ieke Moerdijk, Jaap van Oosten, *Sets, Models, and Proofs*, 2018
- Tom Leinster, *Rethinking set theory*, arXiv:1212.6543
- Todd Trimble, *Fully formal ETCS*, nLab
- Ryszard P. Kostecki, *An Introduction to Topos Theory*, 2011
- Olivia Caramello, *An invitation to topos-theoretic model theory*, 2020

in addition to reading material posted to the Ma6c course webpage