Ma 2a P: Homework N.3

due Tuesday Oct 26, 12 noon

1. Determine if the following functions are linearly independent:
   - $f_1(t) = \cos(2t) - 2\cos^2(t)$ and $f_2(t) = \cos(2t) + 2\sin^2(t)$;
   - $f_1(t) = e^{5t}$ and $f_2(t) = e^{5(t+1)}$.

2. Let $y_1(t)$ and $y_2(t)$ be two solutions of the homogeneous second order equation
$$y'' + p(t)y' + q(t)y = 0$$
where $p(t)$ and $q(t)$ are continuous on an interval $t \in I = (\alpha, \beta)$.
   - If the Wronskian of the two solutions is constant, what can one say about $p(t)$ and $q(t)$?
   - Show that if $y_1(t)$ and $y_2(t)$ vanish at the same point in the interval I, or if they have a maximum or a minimum at the same point, then they are not the fundamental set of solutions.

3. A second order linear homogeneous equation $P(t)y''+Q(t)y'+R(t)y = 0$ is *exact* if it can be written in the form
$$(P(t)y')' + (f(t)y)' = 0,$$
for a function $f(t)$ that depends on $P(t)$, $Q(t)$, and $R(t)$.
   - Write the solutions of an exact equations in terms of $P(t)$, $f(t)$ and integrals.
   - Show that the condition $P''-Q'+R = 0$ is necessary for exactness.

4. If an equation $P(t)y'' + Q(t)y' + R(t)y = 0$ is not exact, one can look for an integrating factor $\mu(t)$ such that $\mu(t)P(t)y'' + \mu(t)Q(t)y' + \mu(t)R(t)y = 0$ is exact.

   - Show that such a factor $\mu(t)$ should satisfy the *adjoint equation*

     $$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$$

     to ensure that the equation $\mu(t)P(t)y''+\mu(t)Q(t)y'+\mu(t)R(t)y = 0$ is exact.

   - Show that the adjoint equation of the adjoint equation is the original equation $Py'' + Qy' + Ry = 0$.

   - Compute the adjoint equation of the Airy equation $y'' - ty = 0$ and of the Bessel equation $t^2y'' + ty' + (t^2 - \nu^2)y = 0$.