1. Determine if the following functions are linearly independent:
   - \( f_1(t) = \cos(2t) - 2\cos^2(t) \) and \( f_2(t) = \cos(2t) + 2\sin^2(t); \)
   - \( f_1(t) = e^{5t} \) and \( f_2(t) = e^{5(t+1)}. \)

2. Let \( y_1(t) \) and \( y_2(t) \) be two solutions of the homogeneous second order equation
   \[ y'' + p(t)y' + q(t)y = 0 \]
   where \( p(t) \) and \( q(t) \) are continuous on an interval \( t \in I = (\alpha, \beta). \)
   - If the Wronskian of the two solutions is constant, what can one say about \( p(t) \) and \( q(t)? \)
   - Show that if \( y_1(t) \) and \( y_2(t) \) vanish at the same point in the interval \( I, \) or if they have a maximum or a minimum at the same point, then they are not the fundamental set of solutions.

3. A second order linear homogeneous equation \( P(t)y'' + Q(t)y' + R(t)y = 0 \) is exact if it can be written in the form
   \[ (P(t)y')' + (f(t)y)' = 0, \]
   for a function \( f(t) \) that depends on \( P(t), Q(t), \) and \( R(t). \)
   - Write the solutions of an exact equations in terms of \( P(t), f(t) \) and integrals.
   - Show that the condition \( P'' - Q' + R = 0 \) is necessary for exactness.
4. If an equation $P(t)y'' + Q(t)y' + R(t)y = 0$ is not exact, one can look for an integrating factor $\mu(t)$ such that $\mu(t)P(t)y'' + \mu(t)Q(t)y' + \mu(t)R(t)y = 0$ is exact.

- Show that such a factor $\mu(t)$ should satisfy the adjoint equation

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$$

to ensure that the equation $\mu(t)P(t)y'' + \mu(t)Q(t)y' + \mu(t)R(t)y = 0$ is exact.

- Show that the adjoint equation of the adjoint equation is the original equation $Py'' + Qy' + Ry = 0$.

- Compute the adjoint equation of the Airy equation $y'' - ty = 0$ and of the Bessel equation $t^2y'' + ty' + (t^2 - \nu^2)y = 0$. 