Euler-Lagrange equations

\[ \frac{\partial L}{\partial q^i} (q,q) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} (q, \dot{q}) \]

\[ = \sum_{j=1}^{n} \frac{\partial^2 L}{\partial q^i \partial q^j} (q, \dot{q}) \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} (q, \dot{q}) \dot{q}^j \dot{q}^j \]

If want to make this solvable for highest derivative

need to invert matrix

\[ H_L (q, \dot{q}) = \left( \frac{\partial^2 L}{\partial q^i \partial q^j} (q, \dot{q}) \right)_{ij} \]

This is the Hessian matrix of \( L \) w.r.t. to the \( q \) coordinates

**Def:** (M,L) Lagrangian system non-degenerate

if in every coord. chart \( H_L \) is invertible

(Note: \( H_L \rightarrow T_{fi}^i T_{l}^f H_L T_{l}^f \) under coord. changes

so invertibility well def.)

Equivalent description of the invertibility condition

1-form \( \Theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} dq^i = \frac{\partial L}{\partial \dot{q}^i} dq^i \)

well def. under coord. charts

(M,L) non-degenerate \iff

2-form \( d\Theta_L \) is non-degenerate
\[ d \Theta_L = \sum_{i_1, \ldots, i_n} \left( \frac{\partial^2 L}{\partial q^{i_1} \partial \dot{q}^{i_1}} d\dot{q}^{i_1} \bigwedge \cdots \bigwedge d\dot{q}^{i_n} \right) \]

non-degen. tata \( d \Theta^m = d \Theta_L \bigwedge \cdots \bigwedge d \Theta_L \) \( n \)-times

2-forms: top forms

= \text{f. vol} \times \text{some function times volume form}

non-deg. if \( f \to \) but if write out what this

\[ d \Theta_L^n \text{ is found det} \left( \frac{\partial^2 L}{\partial q^{i_1} \partial \dot{q}^{i_1}} \right) d\dot{q}^{i_1} \bigwedge \cdots \bigwedge d\dot{q}^{i_n} \]

\[ I = \int_M (\Theta_L) \text{ the Noether integral} \]

\[ I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \]

where \( X = \sum a(q) \frac{\partial}{\partial q} \) vector field of flow of

one-param. symmetries

\( X' \) left to \( TM \) of vect. field \( X \) on \( M \)

\[ X' = \sum a'(q, \dot{q}) \frac{\partial}{\partial q} + \sum b(q, \dot{q}) \frac{\partial}{\partial \dot{q}} \]

\( \forall b' = 0 \ a'(q, \dot{q}) = a(q) \) (because \( \Theta_L \) 1-form \( \in T^*(TM) \) not \( \in T^*(\mathbb{R}^n) \))

Moreover \( X \) being infinitesimal symmetry implies

\[ \mathcal{L}_X (\Theta_L) = 0 \]
$TM$ tangent bundle and $T^*M$ cotangent bundle

$(q, \dot{q})$ coordinates

$p^i = \text{dual basis to the } \frac{\partial}{\partial q^i}, \text{ i.e. } dq^i$

i.e. $p^i(df) = \frac{\partial f}{\partial q^i}$

**Def:** Liouville canonical 1-form on $T^*M$

$$\theta = \sum_{i=1}^{n} p^i dq^i = pdq$$

(again well def under coordinate changes)

invariant definition:

$$v \in T_{(q,p)}(TM) \quad \theta(v) = p(\pi_*(v))$$

$$\pi: T^*M \rightarrow M$$

**Legendre transformation**

$$\tau_L : TM \rightarrow T^*M \quad \text{(fiberwise map)}$$

s.t. $\tau_L^*(\theta) = \theta_L$

$$\tau_L(q, \dot{q}) = (q, p) \quad \text{with } p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q})$$
4. $T_L$ is a local diffeomorphism iff $H_L$ invertible \((M,L)\) nondegenerate

Comment: on any nfold have $TM \& T^*M$ paired by a duality

but cannot "identify" $TM \& T^*M$ unless have extra structure on $M$

example: if $M$ has a Riemannian metric $g_{\mu\nu}$

then can use this to identify (non-canonically) $TM \& T^*M$ (lowering/raising indices)

$g_{\mu\nu} \nu^\mu = \omega_\mu$

This is a special case of the identification via Legandre transform when $L$ has kinetic term given by $g_{\mu\nu}$

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = g_{\mu\nu}(q) \dot{q}^\nu$$

$$L(q,\dot{q}) = g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu$$

Hamiltonian function $H : TM \rightarrow \mathbb{R}$

$$H \circ T_L = E_L = \dot{q} \frac{\partial L}{\partial \dot{q}} - L$$

$$\Rightarrow H(q,p) = (pq - L(q,\dot{q})) \bigg| \frac{\partial L}{\partial \dot{q}}$$
Legendre transform

\[ H = \mathcal{L}(L) \] is invertible (equiv. to convexity assumption in simple one-dim case)

if \( L \) invertible

i.e. \( \frac{\partial L}{\partial q^i} \) invertible

then \( L \) involution

Assume invertibility: (which in phys. terms is an assumption on pos. def. of kinetic energy term in \( L \))

then Euler–Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0
\]

are equivalent to a system of first order equations

\[
\begin{align*}
\dot{p}_i &= -\frac{\partial H}{\partial q^i} \\
\dot{q}_i &= \frac{\partial H}{\partial p_i}
\end{align*}
\]

Hamiltonian equations

\[
\begin{align*}
\frac{dH}{dt} &= \frac{\partial H}{\partial q^i} \cdot \dot{q}^i + \frac{\partial H}{\partial p_i} \cdot \dot{p}_i \\
&= \frac{\partial H}{\partial q^i} \cdot \dot{q}^i + \frac{\partial H}{\partial p_i} \cdot \dot{p}_i \\
&= \left( -\frac{\partial L}{\partial \dot{q}^i} \cdot \dot{q}^i - \frac{\partial L}{\partial q^i} \cdot \dot{p}_i \right) \left|_{p = \frac{\partial L}{\partial \dot{q}^i}} \right.
\end{align*}
\]

\[ \Rightarrow \dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = \frac{\partial H}{\partial q} \]
Conservation of energy:

\[ \frac{dH}{dt} = \frac{\partial H}{\partial q} \frac{dq}{dt} + \frac{\partial H}{\partial p} \frac{dp}{dt} = \frac{\partial H}{\partial q} \frac{dq}{dt} - \frac{\partial H}{\partial p} \frac{dp}{dt} = 0 \]

Given a Hamiltonian \( H: TM^* \rightarrow \mathbb{R} \) 

& corresponding equations

\[ \begin{cases} \dot{q} &= -\frac{\partial H}{\partial p} \\ \dot{p} &= \frac{\partial H}{\partial q} \end{cases} \Rightarrow \text{vector field on } TM^* \]

Hamiltonian vector field

\[ X_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial q_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial p_i} \right) = \frac{\partial H}{\partial q} \frac{\partial}{\partial q} - \frac{\partial H}{\partial p} \frac{\partial}{\partial p} \]

if we integrate this vector field (integral curves exist for all \( t \))

\( \Rightarrow \) Hamiltonian phase flow

\[ \{ q(t), p(t) \} \text{ one-parameter group of } \text{Diff}(TM^*) \]

\[ q(t, (q(0), p(0))) = (q(t), p(t)) \text{ solution with } (q(0), p(0)) = (q_0, p_0) \]
Symplectic form on $T^*\mathbb{R}$

Liouville form $\theta$ on $T^*\mathbb{R}$

$$\theta = p \, dq$$

in local coord's

$\Rightarrow$ $d\theta = \omega$ 2-form in coord's

$$\omega = dp \wedge dq$$

non-degenerate 2-form

$$(d\theta)^n = \text{volume form}$$

$$dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$$

$\omega = \text{canonical symplectic form on } T^*\mathbb{R}$ (in Darboux coordinates)

\underline{Symplectic manifolds} (generalization of $T^*\mathbb{R}$)

A smooth manifold $M$ dim $2n$ (even real dimension)

w/ a closed 2-form $\omega$ ($d\omega = 0$)

which is non-degenerate: $\omega^n \neq 0$ on all $M$

(Nowhere vanishing)

$$\frac{\omega^n}{n!} = \text{Liouville volume form on } M$$

Lagrangian submanifold $L \subseteq M$ if

$$\dim L = \frac{1}{2} \dim M \quad \text{and} \quad \omega|_L = 0$$

Symplectomorphism $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$

Smooth $\varphi : M_1 \to M_2$ with $\omega_1 = (\varphi^* \omega_2)$

\underline{Note:}

Darboux's theorem

$\omega$ on $M$ sympl. can always be written locally as $\omega = dp_1 dq_1$

made people erroneously think sympl. in folds were easy. In fact, very complicated classes of invariants (GW; Floer, etc.)
non-degenerate 2-form $\omega$ on $T^*M$ (or more generally on sympl. manifold $M$)

defines isomorphism $J : T^*(M) \to T(M)$ by

setting

$$\omega(v_1, v_2) = \langle \overline{J}(v_1), v_2 \rangle$$

$v_1 \in T^*_p(M)$

[Pairing by duality of $T^*_M$ & $TM$]

i.e. $\overline{J}(v_1) = \omega(v_1, \cdot)$ (just like pairing done via a Riemann metric)

Hamiltonian vector field is just

$$J(\text{d}H) = X_H$$

Hamiltonian phase flow preserves the symplectic form: symp. morphisms not just diffeomorphisms

enough to check

$$\frac{d}{dt} (\phi^*\omega) \bigg|_{t=0} = \phi^*_H(\omega) = 0$$

$$\phi_*^X(df) = d(X(f)) \quad \text{by def. of } \phi_* \text{ on diff. forms}$$

Check:

$$\mathcal{L}_X(dp_i) = -d\left(\frac{\partial H}{\partial q^i}\right)$$

$$\mathcal{L}_X(df^i) = d\left(\frac{\partial H}{\partial p_i}\right)$$

Then get:

$$\phi^*_X(\omega) = \sum \partial_J(dp^i) \wedge dq^i + dp^i \wedge \partial_J(df^i) = \phi^*_H(\omega)$$
\[
\sum_{i=1}^{n} \left( -d \left( \frac{\partial H}{\partial q_i} \right) \wedge dq_i + dp_i \wedge d \left( \frac{\partial H}{\partial p_i} \right) \right) = -d(H) = 0
\]

\[\Rightarrow \text{ also } \frac{\partial}{\partial t} \text{ Hamiltonian flow preserves Liouville volume form} \]

**Least action principle in phase space**

\[\mathbb{T}M \quad \text{(}q, \dot{q}\text{)}\]
configuration space \( M \)

\[\mathbb{T}^*M \quad \text{(}q, p\text{)}\]
phase space

**Poincaré-Cartan form**

\[
\theta = H dt = p \cdot dq - H dt
\]

(Note: relativistic viewpoint.

\[\langle (p, H), (dq, dt) \rangle\] in Lorentzian metric: Energy = time component of momentum)

\[\text{1-form on } \mathbb{T}^*M \times \mathbb{R}\]

(extended phase space)

**γ: [t₀, t₁] → \mathbb{T}^*M path**

lift \( \sigma \) of \( γ \) to extended phase space

\[
\sigma(t) = (γ(t), t)
\]

admissible paths on \(\mathbb{T}^*M \times \mathbb{R}\)

= lifts of paths on \(\mathbb{T}^*M\)

\[\widetilde{P}(\mathbb{T}^*M)\]

\[\sigma_\varepsilon = \text{ variation of an admissible path by admissible paths w/ fixed ends }\]

\[s_0 = \frac{\partial \sigma_\varepsilon}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \quad \text{infinitesimal variation}\]
\[ S(\sigma) = \int_{\sigma} pdq - H dt = \int_{t_0}^{t_1} (p \dot{q} - H) dt \]

Poincaré's action functional on phase space:
\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} S(\sigma_{\varepsilon}) = 0 \quad \text{critical points} \]

Check: if satisfy Hamiltonian equations

\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} S(\sigma) = \sum_{i=1}^{n} \int_{t_0}^{t_1} \left( \dot{q}^i \delta p_i - \dot{p}_i \delta q^i \right) dt + \sum_{i=1}^{n} p_i \delta q^i \bigg|_{t_0}^{t_1} \]

Integration by parts

\[ \text{using } \delta q^i(t_0) = 0 = \delta q^i(t_1) \text{ get only integral term} \]

\[ \Rightarrow \text{vanishing for arbitrary } \delta p_i \text{ & } \delta q^i \]

\[ \Rightarrow \quad q^i - \frac{\partial H}{\partial p_i} = 0 \]

\[ \dot{p}_i + \frac{\partial H}{\partial q^i} = 0 \]

Note: if left of path \( \gamma \) and \( T_L : TM \rightarrow TM \) invertible

\[ S(\sigma) = \int_{t_0}^{t_1} (p \dot{q} - H) dt = \int_{t_0}^{t_1} L(\gamma'(t), t) dt \]

so same as action functional in configuration space