FINAL

MA109A: FALL 2017

(1) One $\mathbb{N} \times \mathbb{N}$ consider the function

$$d(n,m) = \begin{cases} 1 + \frac{1}{n+m} & m \neq n \\ 0 & n = m. \end{cases}$$

- Show that $d(n,m)$ is a metric on $\mathbb{N}$ and that $(\mathbb{N}, d)$ is complete.
- Show that $(\mathbb{N}, d)$ is not compact by constructing a family of closed sets that does not satisfy the finite intersection condition.

(2) In the unit square $\mathcal{I} \times \mathcal{I}$, with $\mathcal{I} = [0,1]$, construct two connected disjoint sets X and Y such that X contains the points $(0,0)$ and $(1,1)$ and Y contains the points $(0,1)$ and $(1,0)$. Can these sets be path connected?

(3) If X is an n-dimensional topological manifold (locally homeomorphic to $\mathbb{R}^n$) that can be triangulated (by n-dimensional simplexes), the Euler characteristic is defined as $\chi(X) = \sum_{i=0}^{n} (-1)^i \# T_i$ where T_i is the set of i-dimensional simplexes in the triangulation.

- Given a decomposition $X = X_1 \cup X_2$ (also admitting triangulations) show that the Euler characteristic satisfies inclusion-exclusion: $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$.
- Give a formula for the Euler characteristic of topological surfaces with boundary.

(4) A graph G consists of a set of vertices V and a set of edges E with assigned incidence relations (which prescribe the vertices each edge is attached to). The topology on G is given by considering each edge as homeomorphic to an interval with the identifications given by the incidence relations. Use Seifert van Kampen theorem to compute the fundamental group of a connected graph.

(5) Let $K \subset [0,1]$ be the middle third Cantor set. Represent points $x \in K$ as $x = \sum_{n=1}^{\infty} a_n 3^{-n}$ with $a_n \in \{0, 2\}$.

- Show that the maps $f : K \to [0,1]$ and $g : K \to K \times K$ given by

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} a_n 2^{-n}, \qquad g(x) = (g_1(x), g_2(x)) = (\sum_{n=1}^{\infty} a_{2n-1} 3^{-n}, \sum_{n=1}^{\infty} a_{2n} 3^{-n})$$

  are continuous surjections.
- Show that the map $F : K \to [0,1]^2$ given by $F(x) = (f \circ g_1(x), f \circ g_2(x))$ is also a continuous surjection and that it is the restriction to K of a continuous surjection $\tilde{F} : [0,1] \to [0,1]^2$ (a space-filling curve).

(6) **Extra Credit Question**[1] Let $\mathcal{C}([0,1])$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$. Consider the metrics

$$d_\infty(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|, \qquad d_2(f,g) = (\int_0^1 |f(t) - g(t)|^2 \, dt)^{1/2}.$$

Let $B_\infty(0)$ and $B_2(0)$ be the balls of radius one around the zero function $f \equiv 0$ in these two metrics. Show that $B_\infty(0) \subset B_2(0)$ and that the complement of $B_\infty(0)$ is dense in $B_2(0)$.

Date: due: 12/06/2017.

[1] The extra credit question is not needed for this exam score, but it can be used to recover some points in case of a low score in a previous HW or midterm. If you include a solution to this problem, points will be added to your previous assignment with the lowest score.