FINAL EXAM

MA109A: FALL 2021

(1) The symmetric product $s^n(X)$ of a topological space X is the quotient of the product X^n by the action of the symmetric group S_n on X^n by permutations of the coordinates, $s^n(X) = X^n/S_n$.

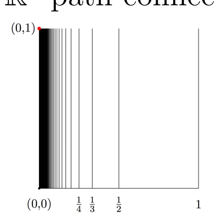
- Show that the sphere S^2 is homeomorphic to the complex projective line $\mathbb{P}^1(\mathbb{C}) = (\mathbb{C}^2 \smallsetminus \{(0,0)\})/\mathbb{C}^*$, with $\lambda \in \mathbb{C}^*$ acting by $\lambda \cdot (z_0, z_1) = (\lambda z_0, \lambda z_1)$.
- Show that the points of the symmetric product $s^n(\mathbb{P}^1(\mathbb{C}))$ can be identified with unordered n-tuples of points in $\mathbb{C} \cup \{\infty\}$, and that an unordered n-tuple can be identified with the set of roots of a monic polynomial $P \in \mathbb{C}[x]$ with $\deg(P) \leq n$, where ∞ is a root if $\deg(P) < n$ (i.e. if $x^n P(1/x)$ vanishes at 0).
- Show that this identification determines a homeomorphism

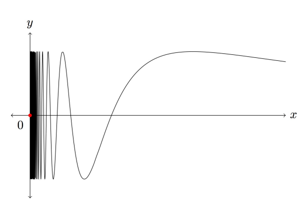
$$s^n(\mathbb{P}^1(\mathbb{C})) \simeq \mathbb{P}^n(\mathbb{C})$$

where the projective space is the quotient $\mathbb{P}^n(\mathbb{C}) = (\mathbb{C}^{n+1} \smallsetminus \{(0,\ldots,0)\})/\mathbb{C}^*$ by the action $\lambda \cdot (z_0, \ldots, z_n) = (\lambda z_0, \ldots, \lambda z_n)$.

(2) Connectedness:

- Consider the space $X = \{0\} \cup \{1/n\}_{n \in \mathbb{N}}$ with the topology induced from the real line. Explain what is the connected component $C(0)$ of the point 0 in X.
- Show that the space $Y = \{(0,1)\} \cup ([0,1] \times \{0\}) \cup \bigcup_{n \in \mathbb{N}} (\{1/n\} \times [0,1])$ with the topology induced from $\mathbb{R}^2$ is connected but not path connected; is the closure $\bar{Y}$ in $\mathbb{R}^2$ path connected, and why?
- Show that the space $Z = \{(0,0)\} \cup \{(x,y) \,|\, y = \sin(1/x),\ x > 0\}$ with the topology induced from $\mathbb{R}^2$ is connected but not path connected; is the closure $\bar{Z}$ in $\mathbb{R}^2$ path connected, and why?





Date: due by end of day Friday, December 10.

(3) Let $\mathcal{C}([0,1])$ be the set of continuous functions $f : [0,1] \to \mathbb{R}$. Consider the metrics

$$d_\infty(f,g) = \sup_{t\in[0,1]} |f(t)-g(t)|, \qquad d_2(f,g) = (\int_0^1 |f(t)-g(t)|^2\, dt)^{1/2}.$$

Let $B_\infty(0)$ and $B_2(0)$ be the balls of radius one around the zero function $f \equiv 0$ in these two metrics. Show that $B_\infty(0) \subset B_2(0)$ and that the complement of $B_\infty(0)$ is dense in $B_2(0)$.

(4) Compute the fundamental group $\pi_1(\Sigma, x_0)$ of a topological surface Σ.

(5) *Every group is the fundamental group of some topological space*:

- Show that the fundamental group $\pi_1(X \vee Y, z_0)$ of the wedge product of two pointed spaces is given by the free product of their fundamental groups
$$\pi_1(X \vee Y, z_0) \simeq \pi_1(X, x_0) \star \pi_1(X, y_0)\,.$$
- Given a group G described in terms of a presentation $G = \langle S \,|\, R\rangle$ with S the set of generators of G and R the set of relations, show that the free group $F = \langle S\rangle$ on the same generators is the fundamental group of a wedge product of a circle for each generator in S,
$$F \simeq \pi_1(\bigvee_S S^1, z_0)\,.$$
- Describe how to modify the space $\bigvee_S S^1$ so as to implement the relations in R and obtain a space with fundamental group G.