HOMEWORK N.7

MA109A: FALL 2021

(1) Let **DirGraphs** be the category of directed graphs, that is, the category of functors $G : \mathbf{2} \to \mathbf{Sets}$ where $\mathbf{2}$ is the category with two objects V, E and two non-identity morphisms $E \overset{s}{\underset{t}{\rightrightarrows}} V$. Let **Cat** be the category of small categories.

- Show that there is a forgetful functor $U : \mathbf{Cat} \to \mathbf{DirGraphs}$.
- Show that this functor admits a left adjoint $F : \mathbf{DirGraphs} \to \mathbf{Cat}$ that associates to a directed graph G the "free category on G" with objects the vertices of G and with non-identity morphisms given by finite directed paths of edges.
- Show that this adjunction induces a monad on **DirGraphs**. For a directed graph G what is the directed graph $UF(G)$ obtained through this monad?

(2) Let **Groupoids** be the category of groupoids.

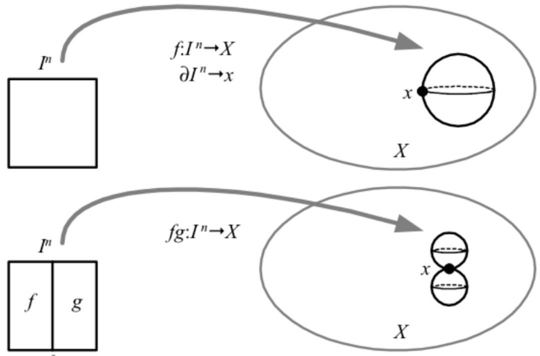
- Show that every category contains a maximal subgroupoid (a subcategory containing all the objects with only invertible morphisms).
- Show that the inclusion $\mathbf{Groupoids} \hookrightarrow \mathbf{Cat}$ has both a left and a right adjoint, where the left adjoint formally adds inverses to morphisms of a category (i.e. it constructs the "category of fractions") and the right adjoint maps a category to its maximal subgroupoid.

(3) *Eckmann–Hilton principle and higher homotopy groups*: Let S be a set. Let $\circ$ and $\bullet$ be two multiplication operations on this set such that both have a unit and they satisfy the relation

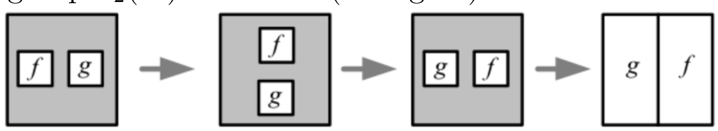
$$(x \bullet y) \circ (z \bullet w) = (x \circ z) \bullet (y \circ w), \qquad \forall x, y, z, w \in S.$$

- Show that the units $1_\bullet$ and $1_\circ$ of the two operations must coincide.
- Show then that both operations are commutative and coincide.
- Show that associativity also holds.
- Given a topological space X and maps $f, g : S^n \to X$, for $n \geq 2$, considered up to homotopy equivalence, show that one can define a product $f \star g : S^n \to X$ by identifying $S^n = \mathcal{I}^n / \sim$ with $\mathcal{I}^n = [0,1]^n$ and the equivalence relation $\sim$ identifying $\partial \mathcal{I}^n$ to a point and composing with the map that contracts the equator of S^n to a point, leaving the one-point-union of two spheres S^n (see figure).

Date: due Monday, December 6, at 2pm.



- In the case $n = 2$ show that the space of maps $f : \mathcal{I}^2 \to X$ admits two operations $\circ$ and $\bullet$ of vertical and horizontal composition (thinking of f as a homotopy between the paths $f(0, \cdot)$ and $f(1, \cdot)$ or as homotopy between the paths $f(\cdot, 0)$ and $f(\cdot, 1)$.
- Use the Eckmann–Hilton argument above applied to these operations to show that the product $\star$ on homotopy classes of maps $S^2 \to X$ is abelian, hence the group $\pi_2(X)$ is abelian (see figure)



- Show that the same argument generalizes to all $n \geq 2$. What goes wrong in the case of $n = 1$?

(4) A (non-directed) graph G consists of a set of vertices V and a set of edges E with assigned incidence relations (which prescribe the vertices each edge is attached to). The topology on G is given by considering each edge as homeomorphic to an interval with the identifications given by the incidence relations. Use Seifert van Kampen theorem to compute the fundamental group of a connected graph.