HOMEWORK N.6

MA109A: FALL 2021

(1) Hauptvermutung for surfaces:
  - Show that any two triangulations T_1 and T_2 of the same topological surface X have a common subdivision.
  - Show that the Euler characteristic $\chi(X,T) = \#V - \#E + \#F$ of a topological surface X with a triangulation T is independent of the choice of triangulation (hence $\chi(X,T) = \chi(X)$ is a homeomorphism invariant of X).

(2) Give a formula for the Euler characteristic of topological surfaces with boundary. Show that orientability, Euler characteristic, and π_0 of the boundary suffice to distinguish all cases of the classification, and that only two out of these three invariants would not suffice.

(3) Show that spheres S^n in any dimension can be triangulated by n-simplexes.

(4) If X is an n-dimensional topological manifold (locally homeomorphic to $\mathbb{R}^n$) that can be triangulated (by n-dimensional simplexes), the Euler characteristic is defined as $\chi(X) = \sum_{i=0}^{n} (-1)^i \# T_i$ where T_i is the set of i-dimensional simplexes in the triangulation.
  - Given a decomposition $X = X_1 \cup X_2$ (also admitting triangulations) show that the Euler characteristic satisfies inclusion-exclusion: $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$.

(5) Let X be a compact metric space.
  - Let f and g be continuous functions from X to S^1 (with the circle viewed as complex numbers of absolute value one, with the induced metric from $\mathbb{C}$). Show that, for sufficiently small δ, if $|f(x) - g(x)| < \delta$, there exists a continuous function $h : X \to \mathbb{R}$ such that $f(x)/g(x) = \exp(ih(x))$. What happens when $|f(x) - g(x)|$ gets larger?
  - Show that a continuous function $f : X \to S^1$ is null-homotopic if and only if there is a continuous $\phi : X \to \mathbb{R}$ with $f(x) = \exp(i\phi(x))$, for all $x \in X$.

(6) Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two given points in $\mathbb{R}^2$. Let $A, B \subset \mathbb{R}^2$ be compact sets neither of which separates x and y (that is, x and y are not contained in different connected components of A^c or of B^c), and such that

Date: due Monday, November 22, at 2pm.

$A \cap B$ is connected. Show that $A \cup B$ also does not separate x and y. (Hint: Borsuk maps and previous problem)

Extended Hint:

- for $K \subset \mathbb{R}^2$ compact, using the fact that connected components of $\mathbb{R}^2 \setminus K$ are path connected, show that x, y are in the same component iff the Borsuk maps $\beta_x|_K$ and $\beta_y|_K$ are homotopic
- Show that two maps $f, g : X \to S^1$ are homotopic iff f/g is null-homotopic
- Consider the ratio $f = \frac{\beta_x|_{A\cup B}}{\beta_y|_{A\cup B}}$ and show that it is null-homotopic