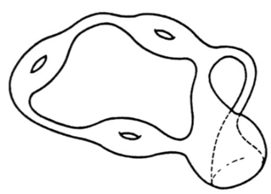
HOMEWORK N.5

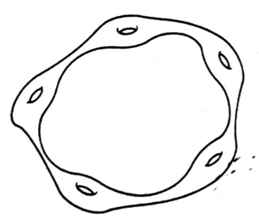
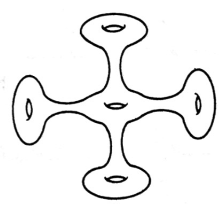
MA109A: FALL 2021

(1) Let X be the space obtained by removing a small disk from a torus $\mathbb{T}^2$ and Y the space obtained by removing three small disks from a sphere S^2. Show X is not homeomorphic to Y but the products $X \times \mathcal{I}$ and $Y \times \mathcal{I}$, with $\mathcal{I} = [0, 1]$, are homeomorphic.

(2) Standard form of topological surfaces:
- Give a standard form presentation of the following topological surfaces, according to the classification theorem:




- Are the following two surfaces homeomorphic? Explain why.

(3) Curves on surfaces:
- Let K be the Klein bottle: find a closed curve γ on K such that cutting along γ produces two Möbius bands.
- Let X be a surface given in standard form by a connected sum of g tori,

$$X = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g-\text{times}}$$

show that there is a collection $\{\gamma_i\}_{i=1}^h$ of pairwise disjoint closed curves on X such that cutting along them gives $X \smallsetminus \cup_i \gamma_i$ a disjoint union of k connected components, each homeomorphic to a "pair of pants" (a sphere S^2 with three small disks removed), with $h = 3(g-1)$ and $k = 2(g-1)$.

Date: due Monday, November 15, at 2pm.

(4) The Stone–Čech compactification is the left adjoint functor of the inclusion functor **CHaus** $\hookrightarrow$ **Top** from the category of compact Hausdorff spaces to the category of topological spaces.

- A *coseparating object* in **CHaus** is an object $\mathcal{I}$ such that for any pair of morphisms $f \neq g$,
$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$
there is a morphism $h : Y \to \mathcal{I}$ such that $h \circ f \neq h \circ y$. Show that the interval $\mathcal{I} = [0,1]$ with the Euclidean topology is a coseparating object. (Note that here the source X can also be in **Top**, while Y is in **CHaus**)
- Given an object X in **Top** the **CHaus**-*under-category* of X is the category $X \downarrow$ **CHaus** whose objects are **Top**-morphisms $f : X \to Y$ with Y in **CHaus** and morphisms $\phi \in \mathrm{Mor}_{X\downarrow\mathbf{CHaus}}(f : X \to Y, f : X \to Y')$ are morphisms $\phi : Y \to Y'$ of **CHaus** that fit in a commutative diagram
$$\begin{array}{ccc} & X & \\ {}^{f}\swarrow & & \searrow^{g} \\ Y & \xrightarrow[\phi]{} & Y' \end{array}$$
Show that the product of all morphisms from X to the coseparating object $\mathcal{I} = [0,1]$ defines an object of $X \downarrow$ **CHaus**
$$X \xrightarrow{\eta} \prod_{\mathrm{Mor}_{\mathbf{Top}}(X,\mathcal{I})} \mathcal{I}\,.$$
- Show that the intersection of all compact Hausdorff subspaces
$$K \subset \prod_{\mathrm{Mor}_{\mathbf{Top}}(X,\mathcal{I})} \mathcal{I}$$
containing the image of η is the closure $\beta(X) := \overline{\eta(X)}$ of the image $\eta(X)$ in $\prod_{\mathrm{Mor}_{\mathbf{Top}}(X,\mathcal{I})} \mathcal{I}$.
- Show the resulting $X \to \beta(X)$ is an initial object of $X \downarrow$ **CHaus**.
- Show that this $\beta(X)$ is the Stone–Čech compactification, by showing that the property above implies that β is a left adjoint to **CHaus** $\hookrightarrow$ **Top**.