HOMEWORK N.4

MA109A: FALL 2021

(1) In $\mathbb{R}^2$ with the standard Euclidean topology consider the equivalence relation $(x,y) \sim (x+n, y+m)$ for all $(n,m) \in \mathbb{Z}^2$.
- Describe the space $Q = \mathbb{R}^2/\sim$ with the quotient topology.
- In $\mathbb{R}^2$ consider a straight line $\ell_\alpha = \{(x,y) \mid y = \alpha x\}$, with γ_α the image of ℓ_α in Q. Show that ℓ_α is closed in $\mathbb{R}^2$ and describe when γ_α is closed in Q.
- In Q consider the relation $[x,y] \sim [x+t, y+\alpha t]$, for $t \in \mathbb{R}$: show that it is an equivalence relation and give a condition on α such that the quotient topological space $Q/\sim$ satisfies the T_1 separation axiom.

(2) Let X be a union of an infinite set Y and two one-point sets $\{x_1\}$ and $\{x_2\}$. Consider the topology $\mathcal{T}$ on X generated by $\mathcal{P}(Y)$ (all subsets of Y) together with any set containing either x_1 or x_2 and all but finitely many points of Y. Show that:
- X is compact,
- X is T_1 but not T_2,
- X is totally disconnected.

(3) The one-point compactification of $(X, \mathcal{T}_X)$ is the set $X^* = X \cup \{\infty\}$, with ∞ an additional point not belonging to X, with the topology

$$\mathcal{T}_{X^*} = \mathcal{T}_X \cup \{(X \smallsetminus K) \cup \{\infty\} \mid K \subseteq X \text{ compact}\}$$

- Show that the topology $\mathcal{T}_{X^*}$ is Hausdorff iff X is Hausdorff and locally compact.
- For X Hausdorff and locally compact, show that the inclusion map $\iota : X \hookrightarrow X^*$ is continuous and open, with $\iota(X)$ open in X^*, and that $\iota(X)$ is dense in X^* iff X is non-compact.
- Describe the space X^* for $X = \mathbb{R}^n$.
- Show that for $X = \mathbb{N}$ it is homeomorphic to $\{0\} \cup \{1/n\}_{n\in\mathbb{N}}$ with the topology induced from the real line.
- Show that a homeomorphism $f : X_1 \to X_2$ of locally compact Hausdorff spaces extends to a homeomorphism of their one-point compactifications.
- Show that the one-point compactification of $\mathbb{Q}$ is not Hausdorff but it is T_1.

Date: due Monday, November 8, at 2pm.

(4) Let X be an infinite set. Let $\mathcal{T}$ be the collection of subsets $U \subseteq X$ with either $U = \emptyset$ or with $X \setminus U$ at most countable.
- Show that $\mathcal{T}$ is a topology on X.
- Give an example of a set X with two inequivalent topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ that have the same convergent sequences.

(5) Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a surjective closed continuous map such that for all $y \in Y$ the preimage $f^{-1}(y)$ is a compact subset of X.
- Show that if X is Hausdorff or regular or locally compact then so is Y.
- Show that if the topology of X has a countable basis then the topology of Y also does.

(6) A Hamel basis is a subset $\{x_\alpha\}$ of $\mathbb{R}$ with the properties that (i) every real number $x \in \mathbb{R}$ can be written as a finite combination $x = \sum_{i=1}^{n} r_i x_{\alpha_i}$ with coefficients $r_i \in \mathbb{Q}$ and (ii) the elements x_α are linearly independent over $\mathbb{Q}$ (that is, $\sum_{i=1}^{n} r_i x_{a_i} = 0$ with $r_i \in \mathbb{Q}$ iff $r_i = 0$ for all i). Show that Hamel bases exist. (Hint: Zorn)