HOMEWORK N.3

MA109A: FALL 2021

(1) Let (X, d) be a metric space. Consider the function

$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that δ is again a metric on X and than d and δ induce the same topology.

(2) Let $\mathbb{Q} = \{r_k\}_{k\in\mathbb{N}}$ be an enumeration of the set of rational numbers. Define a metric on the set $\mathbb{R}$ of real numbers by

$$d_r(x, y) = |x - y| + \sum_{k=1}^{\infty} 2^{-k} \inf\{1, \left|\max_{j\le k} \frac{1}{|x - r_j|} - \max_{j\le k} \frac{1}{|y - r_j|}\right|\}.$$

Show that d_r is a metric on $\mathbb{R}$.

(3) Let $\mathcal{T}$ be the standard Euclidean topology on $\mathbb{R}$ and let $\mathcal{T}_r$ be the topology induced by the metric d_r of the previous problem.

- By comparing local neighborhoods in the two metrics show that $\mathcal{T} \subset \mathcal{T}_r$.
- Show that $\mathbb{Q}$ is an open set in $\mathcal{T}_r$.

(4) Let (X, d) be a metric space. Let $\mathcal{K}$ be the collection of all subsets $A \subseteq X$ that are non-empty, bounded with respect to the metric d, and closed in the topology $\mathcal{T}_d$. Define $f : X \times \mathcal{K} \to \mathbb{R}$ by $f(x, A) = \inf_{y\in A} d(x, y)$ and $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ by $g(A, B) = \sup_{x\in A} f(x, B)$. Show that the Hausdorff distance

$$\delta(A, B) = \max\{g(A, b), g(B, A)\}$$

is a metric on the space $\mathcal{K}$.

(5) Let $X_1 = [-2, 2] \times [-2, 2]$ and $X_2 = X_1 \setminus \{(0, y) \,|\, y \le 0\}$. For $i = 1, 2$ let $\mathcal{W}_i$ be the space of continuous functions on X_i that satisfy the conditions

$$\sup_{(x,y)\in X_i} |f(x, y)| < \infty \quad \text{and} \quad \sup_{(x,y)\in X_i} |\frac{\partial}{\partial x} f(x, y)| < \infty$$

Date: due Monday, October 25, at 2pm.

- Check whether the expression

$$d(f,g) = \sup_{-2\leq y\leq 2} |f(1,y) - g(1,y)| + \sup_{(x,y)\in X_i} |\frac{\partial}{\partial x}(f(x,y) - g(x,y))|$$

  defines a metric on $\mathcal{W}_i$ for either $i=1$ or $i=2$.
- When the answer to the previous question is positive, check whether the resulting metric space is complete.

(6) Show that taking $\mathrm{Obj}(\mathbf{Met}) = \{(X, d_X)\}$ to be all metric spaces and

$$\mathrm{Mor}_{\mathbf{Met}}((X,d_X),(Y,d_Y))$$

to be the "metric maps", namely continuous functions $f : X \to Y$ such that

$$d_Y(f(x), f(y)) \leq d_X(x,y), \quad \forall x, y \in X$$

defines a category **Met** of metric spaces.

- Show that in **Met** monomorphisms are injective metric maps, epimorphisms are metric maps for which the domain has a dense image in the range, and isomorphisms are isometries (distance preserving, injective and surjective).
- Show that the category **Met** is not balanced by giving an explicit example of a morphism in **Met** that is a monomorphism and an epimorphism but not an isomorphism.
- Show that **Met** has an initial and a terminal object and that it has finite products. What goes wrong with infinite products?
- Show that the category **Met** does not have coproducts.