

HOMEWORK N.2

MA109A: FALL 2021

- (1) Let X be a set with an operation $\beta : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ with the properties that
- (i) $\beta(\emptyset) = \emptyset$,
 - (ii) $\beta(A) = \beta(A^c)$, for all $A \in \mathcal{P}(X)$,
 - (iii) $\beta(\beta(A)) \subseteq \beta(A)$, for all $A \in \mathcal{P}(X)$,
 - (iv) $A \cap B \cap \beta(A \cap B) = A \cap B \cap (\beta(A) \cup \beta(B))$, for all $A, B \in \mathcal{P}(X)$.
- Show that the following collection of sets is a topology

$$\mathcal{T} = \{(A \cup \beta(A))^c \mid A \in \mathcal{P}(X)\}$$

- What is the operation $A \mapsto \beta(A)$ in the topology $\mathcal{T}$?
- (2) Let $(X, \sim)$ be a set with an equivalence relation $\sim$ and let $\mathcal{P}$ be the induced partition of X give by the equivalence classes with respect to $\sim$.
- Check that the collection of sets $\mathcal{P} \cup \{\emptyset\}$ is a basis for a topology $\mathcal{T}$.
 - Show that in the topology $\mathcal{T}$ the complement of every open set is still an open set.
- (3) Let $(X, \sim)$ be as in the previous problem. Let $Q = X / \sim$ be the quotient, with the projection map $\pi : X \rightarrow Q$. Consider the collection of subsets of Q given by

$$\mathcal{T}_Q = \{A \subseteq Q \mid \pi^{-1}(A) \in \mathcal{T}\},$$

where $\mathcal{T}$ is the topology constructed in the previous problem.

- Show that $\mathcal{T}_Q$ is a topology on Q .
 - Identify what topology is obtained in this way and explain why.
- (4) In the category **Top** of topological spaces show that the equalizer of the maps

$$\mathbb{R}^2 \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} \mathbb{R}$$

with $f(x, y) = x^2 + y^2$ and $g(x, y)$ the constant function equal to 1 is

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

- (5) Let X be an infinite set. What is the coarsest topology on X in which all infinite subsets of X are open? Explain why.