HOMEWORK N.1

MA109A: FALL 2021

(1) Construct explicitly a bijection $f : (0,1) \to \mathbb{R}$.

(2) Let A be an infinite set and B a countable (finite or countably infinite) set.
- Show that there exists a bijection $f : A \cup B \to A$
- Show that if A is a continuum and $B \subset A$ is countable then $A \smallsetminus B$ is a continuum

(assume validity of the axiom of choice)

(3) Let $A \subset \{0,1\}^{\mathbb{N}}$ be the subset of all sequences that are eventually periodic, i.e. sequences $x = (x_n)$ such that, for some n_0 and k (depending on x), $x_{n+k} = x_n$ for all $n > n_0$.
- Show that the set A is countable
- Construct a bijection $f : \{0,1\}^{\mathbb{N}} \smallsetminus A \to \mathbb{R} \smallsetminus \mathbb{Q}$

(4) Let E be a set and $\{A_n\}_{n\in\mathbb{N}}$ a family of subsets of E. Define the limsup and liminf of the family as

$$\limsup_n A_n = \cap_{n=0}^{\infty}(\cup_{k=0}^{\infty} A_{n+k}), \qquad \liminf_n A_n = \cup_{n=0}^{\infty}(\cap_{k=0}^{\infty} A_{n+k}).$$

- Show that $\limsup_n A_n = \{x \in E \,|\, x \in A_n \text{ for infinitely many } n\}$
- Show that $\liminf_n A_n = \{x \in E \,|\, x \in A_n \text{ for all but finitely many } n\}$
- Show that $\cap_n A_n \subseteq \liminf_n A_n \subseteq \limsup_n A_n \subseteq \cup_n A_n$
- Show that $\liminf_n A_n^c = (\limsup_n A_n)^c$
- Show the following properties:

$$\liminf_n A_n \cup \liminf_n B_n \subseteq \liminf_n (A_n \cup B_n), \qquad \liminf_n A_n \cap \liminf_n B_n = \liminf_n (A_n \cap B_n)$$

$$\limsup_n A_n \cup \limsup_n B_n = \limsup_n (A_n \cup B_n), \qquad \limsup_n A_n \cap \limsup_n B_n \supseteq \limsup_n (A_n \cap B_n)$$

- Show that if $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \cdots$ or $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \cdots$ then $\limsup_n A_n = \liminf_n A_n$

(5) Describe initial object, terminal object, products and coproducts in the categories **Grps** (of groups and group homomorphisms) and $\mathbf{Vec}_k$ (of finite dimensional vector spaces over a field k and linear maps); explain why they satisfy the required properties.

(6)
- Show that the category of finite groups does not have a coproduct.

Date: due: Monday, October 11, 2021, at 2pm.

- Give an example (other than the opposite category of the previous example) of a category without products and explain why.
- Show that the category of fields has no initial and no terminal object.
- Show that the category of fields of characteristic a given prime p has an initial object.
- Show that the category **AbGrps** of abelian groups is a full subcategory of **Grps**, but the coproduct of two abelian groups in **AbGrps** is not the same as their coproduct in **Grps**.

(7) Describe the category $\mathbf{SSet}_X$ of subsets of a given set X; show that the union is the coproduct in this category; explain why the union cannot be the coproduct in **Sets**.

(8) Let $\mathbf{Eucl}_*$ be the category of pointed Euclidean spaces with objects the pairs $(\mathbb{R}^n, x)$ with $x \in \mathbb{R}^n$ (Euclidean spaces with a distinguished vector) and morphisms $f : (\mathbb{R}^n, x) \to (\mathbb{R}^m, y)$ be differentiable functions with $f(x) = y$. Consider the assignment $D : (\mathbb{R}^n, x) \mapsto \mathbb{R}^n$ (forgetting the distinguished point) and $D : f \mapsto Df|_x$ (the derivative seen as the linear map $\mathbb{R}^n \to \mathbb{R}^m$ given by the Jacobian of f). Show this gives a functor $D : \mathbf{Eucl}_* \to \mathbf{Vect}_{\mathbb{R}}$.

(9) Let $(X, \leq)$ be a poset. Show that one can view $(X, \leq)$ as a category with objects the points of X and a morphism $x \to y$ iff $x \leq y$.
- A diagram in a category $\mathcal{C}$ is a functor $F : \mathcal{J} \to \mathcal{C}$ from a small category $\mathcal{J}$. Show that, because between any two objects in $(X, \leq)$ there is at most one morhisms, to construct limits and colimits it suffices to consider diagrams in $(X, \leq)$ determined by the choice of a subset $S \subset X$.
- Show that the limit of the diagram S is the greatest lower bound of S and the colimit is the least upper bound of S.