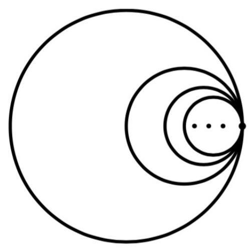
HOMEWORK N.7

MA109A: FALL 2017

(1) **Hawaiian Earrings**: Let $C_n = \{z \in \mathbb{C} : |z - \frac{1}{n}| = \frac{1}{n}\}$. The Hawaiian Earring is the topological space $H = \cup_{n=1}^{\infty} C_n$ with the induced topology from $\mathbb{R}^2$.



- Show that it is possible to construct a continuous function $f : [0,1] \to H$ such that $f(t)$ goes around the circle C_1 for t in the middle third interval of $[0,1]$, around the circle C_2 for t in the middle third of the two remaining intervals, etc. Show that for such a function $f^{-1}(0)$ is the middle third Cantor set in $[0,1]$.
- Show that, for all N, the paths ℓ_i that go around each of the first N circles C_i generate a subgroup of $\pi_1(H)$ isomorphic to the free group F_N. Show that the surjections $H \to H_N = \cup_{n=1}^{N} C_n$ that collapse the remaining curves C_n with $n > N$ to 0 induces a group homomorphism $\phi_N : \pi_1(H) \to F_N$. Show that the surjection $H_N \to H_{N-1}$ that collapses the curve C_N to 0 induces a group homomorphism $\alpha_N : F_N \to F_{N-1}$ that is the identity on the first $N-1$ generators of F_N and kills the N-th one. Using van Kampen theorem show that $\pi_1(H,0)$ decomposes into a free product $F_N \star G_{\geq N+1}$ for some subgroup $G_{\geq N+1}$.
- The projective limit $F = \varprojlim_N F_N$ is the subgroup of the product $\prod_{N=1}^{\infty} F_N$ consisting of elements $g = (g_N)_{N\geq 1}$ satisfying $\alpha_N(g_N) = g_{N-1}$. Show that the maps ϕ_N determine a group homomorphism $\phi : \pi_1(H,0) \to F$.
- Let $\pi \subset F$ be the subgroup of elements $g = (g_k)$ with $g_k \in F_k$ such that for all j the number of times a generator ℓ_j occurs in g_k is bounded in k. Show that $\pi = F_N \star \pi_{\geq N+1}$ with $\pi_{\geq N+1}$ the subgroup given by the element $g = (g_k)$ where the generators ℓ_j with $j \leq N$ do not occur in g_k for all k.
- Show that the image of $\phi : \pi_1(H,0) \to F$ is the subgroup $\pi \subset F$. To show that all elements of π are in the image, given $x \in \pi$ construct a sequence of maps $f_N : [0,1] \to H$, one for each decomposition $F_N \star \pi_{\geq N+1}$, obtained by dividing $\mathcal{I} = [0,1]$ into subintervals so that f_1 realizes the homotopy classes in $F_1 = \pi_1(C_1)$ on the odd subintervals and is trivial on the even one; f_2 is like f_1 on the odd subinterval (realizing the loops around C_1), with the even subintervals further subdivided to realize the loops around C_2, etc. Show that the bound on the number of occurrences of the generators implies that the sequence f_N is uniformly convergent and the resulting limit of these functions (which is similar to the function of the first point of this exercise) realizes the given element of π.

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- Use the decomposition of the fundamental group of H as $F_N \star G_{\geq N+1}$ and the corresponding decomposition of $\pi = F_N \star \pi_{\geq N+1}$ to show that $\phi : \pi_1(H, 0) \to \pi$ is injective, hence an isomorphism.
- Show that π is uncountable. Show that any group homomorphism $g : \pi \to \mathbb{Z}$ such that $g(\ell_i) = 0$ for all i has to map all elements of π to 0, hence the morphism $\mathrm{Hom}(\pi, \mathbb{Z}) \to \prod_{n\geq 1} \mathbb{Z}$ that sends $g : \pi \to \mathbb{Z}$ to $(g(\ell_1), g(\ell_2), \ldots)$ is injective. Use these facts to show that $\pi_1(H)$ is not a free group.

(2) **Symmetric Products of Surfaces**: Let X be an orientable compact topological surface without boundary. Let $s^n(X) = X^n/S_n$ be the n-th symmetric product, the quotient of the Cartesian product X^n by the action of the symmetric group S_n that permutes the factors. Let Γ be the fundamental group of X.

- The semidirect product group $\Gamma^n \rtimes S_n$ is the set of pairs (g, σ) with $g = (g_1, \ldots, g_n) \in \Gamma^n$ and $\sigma \in S_n$ with $(g, \sigma)(h, \tau) = (g\sigma(h), \sigma\tau)$ with $\sigma(h) = (h_{\sigma(1)}, \ldots, h_{\sigma(n)})$. Let H_n be the normal subgroup of $\Gamma^n \rtimes S_n$ generated by the elements of S_n with quotient group $G_n = \Gamma^n \rtimes S_n/H_n$. Give $g \in \Gamma$, let $g^{(i)}$ be the element of Γ^n that has g in the i-th coordinate and the identity element 1 in all the other coordinates. Show that $g^{(i)}h^{(j)} = h^{(j)}g^{(i)}$ for all $i \neq j$ and all $g, h \in \Gamma$ and that for any $g \in \Gamma$ and $i \neq j$ the element $(g^{(i)}, 1) \in \Gamma^n \rtimes S_n$ is the same as the element $(1, \sigma_{ij})^{-1}(g^{(j)}, 1)(1, \sigma_{ij})$ where σ_{ij} is the permutation that exchanges the i-th and j-th coordinates. Show that the quotient group G_n is isomorphic to the abelianization $\Gamma^{ab} = \Gamma/[\Gamma, \Gamma]$.
- Let $\Delta \subset X^n$ be the set of points of X^n where at least two of the coordinates are equal. Show that the action of S_n is free on $X^n \smallsetminus \Delta$. Show that the fundamental group of the quotient $B_n(X) = (X^n \smallsetminus \Delta)/S_n$ is equal to $\pi_1(X^n \smallsetminus \Delta) \rtimes S_n$ by showing that covering spaces of $B_n(X)$ can be equivalently described as S_n-equivariant covering spaces of $X^n \smallsetminus \Delta$, namely covering spaces $\pi : E_\Delta \to (X^n \smallsetminus \Delta)$ with an action $\tilde{\sigma}$ of S_n on E_Δ satisfying $\sigma \circ \pi = \pi \circ \tilde{\sigma}$. Assuming the fact that the fundamental group $\pi_1(X^n \smallsetminus \Delta)$ maps surjectively to Γ^n, identify a condition on the action of H_n that determines when a covering space of $B_n(X)$ extends to a covering space of $s^n(X)$.
- Using the previous steps show that, for all $n > 1$, the fundamental group of $s^n(X)$ is isomorphic to the abelianization $\pi_1(X)^{ab}$.

(3) **Symmetric products of the sphere**: The complex projective space $\mathbb{P}^n(\mathbb{C})$ is the quotient of the set of nonzero vectors $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by the equivalence relation $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for λ any non-zero complex number, with the quotient topology.

- Show that $\mathbb{P}^1(\mathbb{C})$ is a topological surface homeomorphic to S^2, by describing $\mathbb{P}^1(\mathbb{C})$ as the union of two open sets $\pi(U_i)$ given by the images under the quotient map of the open sets $U_i = \{(z_0, z_1) \,|\, z_i \neq 0\}$, and that $\mathbb{P}^1(\mathbb{C}) = \pi(U_0) \cup \{\pi(0, 1)\}$ is homeomorphic to the one-point compactification of $\mathbb{C}$.
- Show that the points of the symmetric product $s^n(\mathbb{P}^1(\mathbb{C}))$ can be identified with unordered n-tuples of points in $\mathbb{C} \cup \{\infty\}$, and that an unordered n-tuple can be identified with the set of roots of a monic polynomial $P \in \mathbb{C}[x]$ with $\deg(P) \leq n$, where ∞ is a root if $\deg(P) < n$ (i.e. if $x^n P(1/x)$ vanishes at 0).
- Show that this identification determines a homeomorphism $s^n(\mathbb{P}^1(\mathbb{C})) \simeq \mathbb{P}^n(\mathbb{C})$. What is the fundamental group of $\mathbb{P}^n(\mathbb{C})$?