# HOMEWORK N.2

MA109A: FALL 2017

(1) Let $(X, \sim)$ be a set with an equivalence relation $\sim$ and let $\mathcal{P}$ be the induced partition of $X$ give by the equivalence classes with respect to $\sim$.
- Check that the collection of sets $\mathcal{P} \cup \{\emptyset\}$ is a basis for a topology $\mathcal{T}$.
- Show that in the topology $\mathcal{T}$ the complement of every open set is still an open set.

(2) Let $(X, \sim)$ be as in the previous problem. Let $Q = X/\sim$ be the quotient, with the projection map $\pi : X \to Q$. Consider the collection of subsets of $Q$ given by
$$\mathcal{T}_Q = \{A \subseteq Q \,|\, \pi^{-1}(A) \in \mathcal{T}\},$$
where $\mathcal{T}$ is the topology constructed in the previous problem.
- Show that $\mathcal{T}_Q$ is a topology on $Q$.
- Identify what topology is obtained in this way and explain why.

(3) Let $X$ be an infinite set. What is the coarsest topology on $X$ in which all infinite subsets of $X$ are open? Explain why.

(4) Let $(X, d)$ be a metric space. Consider the function
$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$
Show that $\delta$ is again a metric on $X$ and than $d$ and $\delta$ induce the same topology.

(5) Let $\mathbb{Q} = \{r_k\}_{k \in \mathbb{N}}$ be an enumeration of the set of rational numbers. Define a metric on the set $\mathbb{R}$ of real numbers by
$$d_r(x, y) = |x - y| + \sum_{k=1}^{\infty} 2^{-k} \inf\{1, \left| \max_{j \leq k} \frac{1}{|x - r_j|} - \max_{j \leq k} \frac{1}{|y - r_j|} \right| \}.$$
Show that $d_r$ is a metric on $\mathbb{R}$.

(6) Let $\mathcal{T}$ be the standard Euclidean topology on $\mathbb{R}$ and let $\mathcal{T}_r$ be the topology induced by the metric $d_r$ of the previous problem.
- By comparing local neighborhoods in the two metrics show that $\mathcal{T} \subset \mathcal{T}_r$.
- Show that $\mathbb{Q}$ is an open set in $\mathcal{T}_r$.

*Date*: due: 10/09/2017.