Complete Metric Spaces

(X, d) $\quad \{x_n\} \subseteq X$ sequence of pts is <u>Cauchy</u> if

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall n, m \geq N$

$d(x_n, x_m) < \varepsilon$

<u>Def</u>: (X, d) metric space is <u>complete</u> if all Cauchy sequences converge

i.e. if $\{x_n\}$ Cauchy $\Rightarrow \exists x \in X$ s.t. $x_n \xrightarrow{d} x$

<u>Note</u>: enough to know every Cauchy sequence has a convergent subsequence

in fact if $x_{n_k} \to x$ $\quad \forall n_k \geq N \quad d(x_{n_k}, x) < \frac{\varepsilon}{2}$

but then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

if N large enough that $d(x_n, x_{n_k}) < \frac{\varepsilon}{2}$ $\forall n, n_k \geq N$

hence the whole sequence converges to same x

<u>Examples</u>: $(\mathbb{R}, d(x,y) = |x-y|)$ complete but

$(\mathbb{Q}, d(x,y) = |x-y|)$ not complete

(construction of real numbers as completion of $\mathbb{Q}$)

<u>Product spaces</u> $X = \prod_\alpha X_\alpha$ $\quad x_n = (x_{n,\alpha})$ sequence

then $x_n \to x$ iff $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α

$\Rightarrow$ because π_α continuous

$\Leftarrow$ take $\mathcal{U}^{(x)} = U^{(x)}_{\alpha_1} \times \cdots \times U^{(x)}_{\alpha_N} \times \prod_{\alpha \notin \{\alpha_i\}} X_\alpha$ $\quad$ for each α_i know $\pi_{\alpha_i}(x_n) \in \mathcal{U}_{\alpha_i}(x_\alpha)$ for $n \geq N_{\alpha_i}$

for $N = \max_i \{N_{\alpha_i}\}$ $\quad x_n \in \mathcal{U}(x)$

$(X,d) \rightsquigarrow$ bounded metric $d'(x,y) = \min\{d(x,y), 1\}$

Note: (X,d) complete iff (X,d') complete
(same small balls; same conv. sequences)

$X^J = \prod_{j\in J} X$ product of copies of X indexed by J

$$\delta(\underline{x},\underline{y}) := \sup_{j\in J}\{d'(x_j, y_j)\}$$

$\underline{x} = (x_j)$
$\underline{y} = (y_j)$

this δ is a metric on X^J
"uniform metric"

if (X,d) complete $\Rightarrow (X^J, \delta)$ also complete

$d'(x_j, y_j) \leq \delta(\underline{x}, \underline{y})$ so given $\underline{x}_n = (x_{j,n})$ sequence in X^J

$d'(x_{j,n}, x_{j,m}) \leq \delta(\underline{x}_n, \underline{x}_m)$

so if $\underline{x}_n$ is Cauchy in (X^J, δ) then
$x_{j,n}$ is Cauchy in the j-th copy of (X, d')

$\Rightarrow x_{j,n} \xrightarrow{d'} x_j$ in X

then $\underline{x} = (x_j) \in X^J$ with $\underline{x}_n \xrightarrow{\delta} \underline{x}$ in X^J

indeed have:
$\forall n \geq N$ ← note N indep of j (Cauchy on X^J)
m

$\forall n \geq N$

$d'(x_{n,j}, x_{m,j}) < \frac{\varepsilon}{2} \Rightarrow d'(x_{n,j}, x_j) \leq \frac{\varepsilon}{2}$ for all j

$\Rightarrow \delta(\underline{x}_n, \underline{x}) \leq \frac{\varepsilon}{2} < \varepsilon$