Metric Spaces (X, d)

$d : X \times X \longrightarrow \mathbb{R}$ function

1) $d(x,y) \geq 0 \quad \forall x,y \in X$ and $d(x,y) = 0$ iff $x = y$
positivity

2) $d(x,y) = d(y,x) \quad \forall x,y \in X$ symmetry

3) triangle inequality

$$d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$$

metric balls
radius $\varepsilon > 0$

$$B_\varepsilon(x) = \{ y \in X \mid d(x,y) < \varepsilon \}$$

$\qquad \qquad \qquad \qquad \qquad \parallel$

or $B_d(x, \varepsilon)$ notation

$$\mathcal{B} = \{ B_d(x,\varepsilon) \mid x \in X, \ \varepsilon > 0 \}$$

basis for a topology $\mathcal{T}_d$

metric topology of (X,d)

– check that $\forall y \in B_d(x,\varepsilon)$

$\exists \ \varepsilon' > 0$ s.t $\underset{\underset{y}{\Psi}}{B_d(y,\varepsilon')} \subseteq B_d(x,\varepsilon)$

i.e. check

if $z \in B_d(y,\varepsilon') \Rightarrow z \in B_d(x,\varepsilon)$

$\Updownarrow \qquad \qquad \qquad \qquad \Updownarrow$

$d(z,y) < \varepsilon' \qquad \qquad d(z,x) < \varepsilon$

$d(z,x) \leq \underset{\varepsilon'}{\underset{\vee}{d(z,y)}} + \underset{\varepsilon}{\underset{\vee}{d(y,x)}}$

enough to choose ε' s.t. $\boxed{\varepsilon' < \varepsilon - d(x,y)}$

- if $B_1 = B_d(x_1, \varepsilon_1)$ and $B_2 = B_d(x_2, \varepsilon_2)$ are two basis elements and $y \in B_1 \cap B_2$ then check $\exists$ $\underset{\underset{y}{\Psi}}{B_d(y, \delta)} \subseteq B_1 \cap B_2$

same idea:

$$d(y, x_1) < \varepsilon_1 \quad d(y, x_2) < \varepsilon_2 \quad \text{because } y \in B_1 \cap B_2$$

take $z \in B_d(y, \delta)$

$$d(z, y) \le d(z, x_1) + d(x_1, y) < \varepsilon_1 \quad \text{if } \delta < \varepsilon_1 - d(x_1, y)$$

$$\le d(z, x_2) + d(x_2, y) < \varepsilon_2 \quad \text{if } \delta < \varepsilon_2 - d(x_2, y)$$

so if $\delta < \min\{\varepsilon_1 - d(x_1, y), \varepsilon_2 - d(x_2, y)\}$
then $z \in B_d(y, \delta) \Rightarrow z \in B_1 \cap B_2$

Example: X set metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \Rightarrow \text{induces discrete topology}$$

$B_d(x, 1) = \{x\}$
or any $\varepsilon \le 1$
so all pts are open sets

Example $X = \mathbb{R}$

$d(x,y) = |x - y|$ Euclidean topology

$(X, \mathcal{T})$ topological space **metrizable** iff $\exists$ d metric on X s.t. $\underset{\text{metric topology}}{\mathcal{T} = \mathcal{T}_d}$ $\boxed{\text{NOT always}}$

metrization theorems

- diameter (X,d)

  $A \subseteq X$ subset **bounded** if $\exists M > 0$

  s.t. $d(x,y) \leq M \quad \forall x,y \in A$

  $A \neq \emptyset \quad diam(A) = \sup\{d(x,y) \mid x,y \in A\}$

- given (X,d) $\quad \exists d'$ metric $\quad$ s.t. (X,d') bounded (all subsets bounded)

  $d'(x,y) = \min\{d(x,y), 1\}$ $\quad$ still a metric

  $d'(x,z) \leq d'(x,y) + d'(y,z)$

  if either one is $= 1$ then verified $\leq$

  if both ≤ 1 then $= d(x,y) + d(y,z)$ so $\geq d(x,z)$ because metric d

same open balls for all radii $\varepsilon \leq 1$

Note: can always form a basis for topology $\mathcal{T}_d$ by taking only balls w/ radius $\varepsilon \leq \varepsilon_0$ (same properties of basis satisfied)

- finer/coarser topologies

  $(X,d) \qquad (X,d')$

  $\mathcal{T}_d \subseteq \underset{\text{finer}}{\mathcal{T}_{d'}}$ iff $\forall x \in X \ \forall \varepsilon > 0$ $\exists \delta > 0: B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$

- if $\mathcal{T}'$ finer than $\mathcal{T}$:

  if $B_d(x,\varepsilon)$ basis element for $\mathcal{T}$

  $\exists$ basis element B' of $\mathcal{T}'$: $x \in B' \subseteq B_d(x,\varepsilon)$

  if $\mathcal{T}' = \mathcal{T}_{d'}$ then $B' \supseteq B_{d'}(x,\delta)$ for some $\delta > 0$

- if have the ε-δ inclusion $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$

  then $\forall B \in \mathcal{B}_d$ $\forall x \in B$ $\exists \varepsilon > 0$ $\underset{\substack{\psi \\ x}}{B_d(x,\varepsilon)} \subseteq B$

  hence $B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon) \subseteq B$

  can fit a basis element of $\mathcal{T}_{d'}$ around each pt inside a basis element of $\mathcal{T}_d$

  so $\mathcal{T}_{d'}$ is finer than $\mathcal{T}_d$

Continuity in metric spaces

(X, d_X) $\qquad$ (Y, d_Y)

$f : X \to Y$ continuous iff

$\forall x \in X$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

- if f continuous $\forall U \in \mathcal{T}_{d_Y}$ $f^{-1}(U) \in \mathcal{T}_{d_X}$

  this means for all $B_{d_Y}(f(x),\varepsilon)$ $\quad f^{-1}(B_{d_Y}(f(x),\varepsilon)) \in \mathcal{T}_{d_X}$

$f^{-1}(B_{d_Y}(f(x),\varepsilon)) \in \mathcal{T}_{d_X}$ means

$\forall y \in \quad \exists B_{d_X}(y,\delta)$ s.t. $B_{d_X}(y,\delta) \subset f^{-1}(B_{d_Y}(f(x),\varepsilon))$

which means:

$\forall z \in B_{d_X}(y,\delta) \quad f(z) \in B_{d_Y}(f(x),\varepsilon)$

i.e. $d_X(z,y) < \delta \Rightarrow d_Y(f(z), f(x)) < \varepsilon$

in particular take $y = x \in f^{-1}(B_{d_Y}(f(x),\varepsilon))$

$\Rightarrow \quad d_X(x,z) < \delta \Rightarrow d_Y(f(x), f(z)) < \varepsilon$

– conversely: if $\quad d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \quad (*)$

$\delta = \delta(\varepsilon)$

given $\mathcal{U} \in \mathcal{T}_{d_Y}$ and $f^{-1}(\mathcal{U}) \subseteq X$ (want to show this is open in X)

take $x \in f^{-1}(\mathcal{U})$: because $f(x) \in \mathcal{U}$ $\exists$ ball $B_{d_Y}(f(x),\varepsilon) \subseteq \mathcal{U}$

by $(*)$ then $\exists B_{d_X}(x,\delta)$ s.t.

$f(B_{d_X}(x,\delta)) \subseteq B_{d_Y}(f(x),\varepsilon))$

$\Rightarrow \quad \underset{x}{\underset{\cup}{B_{d_X}(x,\delta)}} \subseteq f^{-1}(\mathcal{U}) \quad \Rightarrow \quad f^{-1}(\mathcal{U})$ open

Convergence characterized by sequences in metric spaces

- convergence in topological sense $(X, \mathcal{J})$

  $A \subseteq X$ set   adherence pts of A

  $$\{x \in X \mid \mathcal{U} \cap A \neq \emptyset \ \ \forall\, \mathcal{U} \in \mathcal{J} \text{ s.t. } x \in \mathcal{U}\}$$

- convergence in metric sense (X, d)

  $x_n \in X$ $x_n \longrightarrow x \in X$ iff $d(x_n, x) \longrightarrow 0$ in $\mathbb{R}$

- Relation between these notions of convergence:

  $x_n \in X$ $x_n \xrightarrow{top} x$ means $\forall\, \mathcal{U} \in \mathcal{J}$, $x \in \mathcal{U}$

  $\exists\, n_0$ s.t. $\forall\, n \geq n_0$ $x_n \in \mathcal{U}$

  in particular if $\mathcal{J} = \mathcal{J}_d$

  $\forall\, B_d(x, \varepsilon)$ $\exists\, n_0$ s.t. $\forall\, n \geq n_0$

  $x_n \in B_d(x, \varepsilon)$ hence $d(x_n, x) < \varepsilon$

$A \subseteq X$ $(X, \mathcal{J})$

1) if $\exists\, x_n \in A$ s.t. $x_n \xrightarrow{top} x \implies x \in \overline{A}$

2) if $\mathcal{J} = \mathcal{J}_d$ metric (X, d) then $x \in \overline{A} \implies \exists\, \underset{\overset{\cap}{A}}{x_n} \xrightarrow{d} x$