Topology $\boxed{4}$

- cluster points: $A \subseteq X$ $x \in X$ is cluster pt of A iff each neighborhood $\mathcal{U}(x)$ contains at least one pt ~~of~~ $y \in A$ with $y \neq x$

accumulation pts

$A' =$ set of accumulation/cluster pts of A

$$= \{ x \in X \mid \forall \mathcal{U}(x) \quad \mathcal{U}(x) \cap (A \setminus \{x\}) \neq \emptyset \}$$

- $\overline{A} = A \cup A'$ closure is union w/ set of accumulation pts

- interior $\mathrm{Int}(A) = \mathring{A}$ largest open set contained in A

$$\mathrm{Int}(A) = \bigcup \{ \mathcal{U} \in \mathcal{J} \mid \mathcal{U} \subseteq A \}$$

- $\mathrm{Int}(A) = \left(\overline{A^c} \right)^c$ hence A open iff $A = \mathring{A}$ ($= \mathrm{Int}(A)$)

since

$$\mathrm{Int}(A) = \bigcup \{ C^c \mid C \in \mathcal{C} \quad C \supseteq A^c \}$$

$$= \Big(\underbrace{\bigcap \{ C \in \mathcal{C} \mid C \supseteq A^c \}}_{\overline{A^c}} \Big)^c$$

$$= \left(\overline{A^c} \right)^c$$

<u>Note</u> interior does <u>not</u> behave well w/ resp to unions:

e.g. $A = \mathbb{Q}$ $B = \mathbb{R} \setminus \mathbb{Q}$ $\mathrm{Int}(A) = \mathrm{Int}(B) = \emptyset$

but $A \cup B = \mathbb{R}$ $\mathrm{Int}(\mathbb{R}) = \mathbb{R}$

- Boundary of A : part of closure that is not in interior

$$\partial A = \bar{A} \setminus \mathring{A}$$

- dense subset $B \subset C$ dense in C if $\bar{B} = C$

- complement of a dense $B \subset X$ has empty interior $\operatorname{Int}(B^c) = \emptyset$

Constructing topologies by assigned properties

<u>closure</u> : X set $u : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ an operation

s.t.
1. $u(\emptyset) = \emptyset$
2. $A \subset u(A) \quad \forall A \in \mathcal{P}(X)$
3. $u(u(A)) = u(A) \quad \forall A \in \mathcal{P}(X)$

$A \subseteq B$
$\Rightarrow u(A) \subseteq u(B)$
since $A \subseteq B$ iff $B = A \cup B$ &

4. $u(A \cup B) = u(A) \cup u(B) \quad \forall A, B \in \mathcal{P}(X)$

$\Rightarrow \exists\, \mathcal{T}$ topology on X s.t. $u(A) = \bar{A}$

it is given by

$$\mathcal{T} = \{\, u(A)^c \mid A \in \mathcal{P}(X) \,\}$$

Verify that this is indeed a topology :

- $\emptyset, X \in \mathcal{T} \qquad \emptyset = u(X)^c = X^c \qquad X = \emptyset^c = u(\emptyset)^c$

- $u(A_1)^c \cap u(A_2)^c = (u(A_1) \cup u(A_2))^c = u(A_1 \cup A_2)^c$
  same for arbitrary <u>finite</u> intersections

- arbitrary unions: $S = \bigcup_\alpha u(A_\alpha)^c$

$$= \left(\bigcap_\alpha u(A_\alpha)\right)^c \quad \text{hence } S^c = \bigcap_\alpha u(A_\alpha) \subseteq u(A_\alpha) \ \forall \alpha$$

$$\Rightarrow u(S^c) \underset{(4)}{\subseteq} u(u(A_\alpha)) \underset{(3)}{=} u(A_\alpha)$$

$$\Rightarrow u(S^c) \subseteq \bigcap_\alpha u(A_\alpha) = S^c$$

$$\overset{(2)}{\Rightarrow} S^c \subseteq u(S^c) \text{ and } u(S^c) \subseteq S^c \Rightarrow S^c = u(S^c)$$

$$\Rightarrow S \in \mathcal{T}$$

Similar facts: $\gamma : P(X) \longrightarrow P(X)$ s.t.

1. $\gamma(\emptyset) = \emptyset$
2. $\gamma(\gamma(A)) \subseteq A \cup \gamma(A)$
3. $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$
4. $\forall x \in X \quad x \notin \gamma(x)$

$\exists\, \mathcal{T} = \{(A \cup \gamma(A))^c,\ A \in P(X)\}$ topology s.t. $\gamma(A) = A'$

accumulation pts

$\eta : P(X) \to P(X)$ interior

1. $\eta(X) = X$
2. $\eta(A) \subseteq A$
3. $\eta(\eta(A)) = \eta(A)$
4. $\eta(A \cap B) = \eta(A) \cap \eta(B)$

$\exists\, \mathcal{T} = \{\eta(A) \mid A \in P(X)\}$ s.t. $\mathrm{Int}(A) = \eta(A)$

etc. (boundary) ~~[illegible]~~