Continuous maps: $(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$

two topological spaces

$f: X \to Y$ a function

induced function (preimage) $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$

Def: $f: X \to Y$ continuous iff

$\forall\, U_Y \in \mathcal{T}_Y \quad f^{-1}(U_Y) \in \mathcal{T}_X$

the preimage of an open set is an open set

Note: for $Y \subset X$: $\mathcal{T}_Y$ the induced topology from $\mathcal{T}_X$ is the smallest topology on Y such that the inclusion map $i: Y \hookrightarrow X$ is continuous

Properties:

- composition of continuous functions is continuous
- restriction is continuous (composition w/ inclusion)
  $f|_A = f \circ i_A \qquad i_A : A \hookrightarrow X$
- continuous to $\text{Range}(f) \subset Y$ w/ induced topology

Category $\underline{\mathrm{Top}}$ of topological spaces

$$\mathrm{Obj}(\underline{\mathrm{Top}}) = \{ (X, \mathcal{T}) \text{ topological spaces} \}$$

$$\mathrm{Mor}_{\underline{\mathrm{Top}}}((X,\mathcal{T}_X),(Y,\mathcal{T}_Y))$$

$$= \{ f: X \to Y \mid f \text{ continuous w/ respect to the topologies } \mathcal{T}_X, \mathcal{T}_Y \}$$

$$\forall \mathcal{U} \in \mathcal{T}_Y \quad f^{-1}(\mathcal{U}) \in \mathcal{T}_X$$

- equivalent def. of continuity:
  preimage of closed sets is closed

  $\forall C \in \mathcal{C}_Y \quad f^{-1}(C) \in \mathcal{C}_X$

> Homeomorphism:
> $f: X \to Y$ bijection
> continuous w/
> continuous inverse

Why this agrees w/ usual ε-δ continuity on $\mathbb{R}$ w/ Euclidean top. ?

$f: \mathbb{R} \to \mathbb{R}$

f continuous at $x_0 \in \mathbb{R}$ if $\forall \varepsilon \ \exists \delta$ s.t.

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

i.e. $\forall$ open $\mathcal{U}_\varepsilon(f(x_0))$ (basis of top. on target)

$\exists\, \mathcal{U}_\delta(x_0)$ s.t. $\mathcal{U}_\delta(x_0) \subseteq f^{-1}(\mathcal{U}_\varepsilon(f(x_0)))$

i.e. $\forall$ open $\mathcal{U}_\varepsilon(f(x_0))$ the preimage $f^{-1}(\mathcal{U}_\varepsilon(f(x_0)))$ is also open

more generally $\mathcal{U}$ in target top
union of such

preimage union of contains open sets of basis of source top. around each pt.

Induced Topology / Subspace topology

$(X, \mathcal{T}_X)$ top. space $Y \subset X$ any subset

$$\mathcal{T}_Y := \{ U \cap Y \mid U \in \mathcal{T}_X \}$$

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special case of inclusion

$Y \hookrightarrow X$ $(X, \mathcal{T}_X)$

more generally can ~~con~~ consider some set S and map of sets

$f : S \to X$ what is "optimal" topology on S that makes f continuous?

minimal / coarsest

$\mathcal{T}_{S,f}$ = inters. of all topologies on S that make f continuous

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$\mathcal{T}_Y$ for $Y \overset{j}{\hookrightarrow} X$ is coarsest topology that makes j inclusion continuous

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Universal property of the induced / subspace topology

characterizing which functions into Y are continuous

Subspace topology $\mathcal{T}_Y$ is characterized by property:

$\forall$ top. space $(Z, \mathcal{T}_Z)$ and any $f : Z \to Y$ funct of sets

f is continuous iff composite $j \circ f : Z \to X$ is continuous

$$\begin{tikzcd}
(Z, \mathcal{T}_Z) \arrow[r, "f"] \arrow[d, "j \circ f"'] & (Y, \mathcal{T}_Y) \arrow[dl, "j"] \\
(X, \mathcal{T}_X) &
\end{tikzcd}$$

- first check $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}_X\}$ satisfies this univ. property

— given $f : (Z, \mathcal{T}_Z) \to Y$ s.t. $j \circ f : (Z, \mathcal{T}_Z) \to (X, \mathcal{T}_X)$ continuous

$\Rightarrow f : (Z, \mathcal{T}_Z) \to (Y, \mathcal{T}_Y)$ cont.

Know that $(j \circ f)^{-1}(U) \in \mathcal{T}_Z$ for all $U \in \mathcal{T}_X$

$\| \quad$ the open subsets of $\mathcal{T}_Y$

$f^{-1}(j^{-1}(U))$ with $j^{-1}(U) = U \cap Y$ so $\forall$ $U \cap Y$ $f^{-1}(U \cap Y) \in \mathcal{T}_Z$

so OK: $f : (X, \mathcal{T}_Z) \to (Y, \mathcal{T}_Y)$ cont.

— if $f : (Z, \mathcal{T}_Z) \to (Y, \mathcal{T}_Y)$ cont. then $j \circ f$ also cont. also OK because j cont w/ resp to $\mathcal{T}_Y$ so comp of conts.

- second: if $\mathcal{T}$ a topology on Y that satisfies univ. prop then $\mathcal{T} = \mathcal{T}_Y$ (i.e. $\mathcal{T} \subseteq \mathcal{T}_Y$ and $\mathcal{T}_Y \subseteq \mathcal{T}$)

since $\mathcal{T}$ has univ. prop. take $(Z, \mathcal{T}_Z) = (Y, \mathcal{T}_Y)$

then for $f = \text{identity} : Y \to Y$

$$\begin{tikzcd}
(Y, \mathcal{T}_Y) \arrow[r, "id"] \arrow[dr, "j"'] & (Y, \mathcal{T}) \arrow[d, "j"] \\
& (X, \mathcal{T}_X)
\end{tikzcd}$$

know j cont. $\Rightarrow$ id cont. so $\mathcal{T}' \subseteq \mathcal{T}_Y$

for $(Z, \mathcal{T}_Z) = (Y, \mathcal{T}')$, $f = id : Y \to Y$ $\mathcal{T}_Y \subseteq \mathcal{T}'$

$(Y, \mathcal{T}') \xrightarrow{id} (Y, \mathcal{T}')$, $(X, \mathcal{T}_X) \hookleftarrow (Y, \mathcal{T}')$

f cont $\Rightarrow$ j cont $\Rightarrow$ $\mathcal{T}_Y \subseteq \mathcal{T}'$

Product topology

$X = \prod_{\alpha \in I} X_\alpha$ as set $\quad (X_\alpha, T_\alpha)$ top. spaces

- $\mathcal{T}_X$ = topology generated by basis
$$\mathcal{B} = \left\{ \prod_{\alpha \in I} U_\alpha \;\middle|\; U_\alpha \in \mathcal{T}_\alpha \text{ and for all but fin. many } \alpha \;\; U_\alpha = X \right\}$$

- projection maps $\pi_\alpha : X \to X_\alpha$

  want minimal topology (coarsest) that makes all these π_α continuous

  (inters. of all topologies that make π_α conts)

- universal property of the product topology:

  for every topological space $(Z, \mathcal{T}_Z)$
  and every function (of sets) $f : Z \to X$

  f is continuous iff $\forall \alpha \in I$
  $$\pi_\alpha \circ f : (Z, \mathcal{T}_Z) \to (X_\alpha, \mathcal{T}_\alpha)$$
  is continuous

Quotient topology

$(X, \mathcal{T}_X)$ and $\sim$ equivalence relation on set X
with quotient $X/\sim$
and projection $\pi : X \to X/\sim$
$x \mapsto [x]$

More generally
S set and surjective map of sets $X \xrightarrow[\pi]{} S$
(can think of S as quotient by identif. $x \sim y$ if $\pi(x) = \pi(y)$)
$S \simeq X/\sim$ $\sim$ gen by

construct a topology on $S = X/\sim$

$$\mathcal{T}_S = \{ U \subseteq S \mid \pi^{-1}(U) \in \mathcal{T}_X \}$$

- finest (largest) topology on $X/\sim$ that makes π continuous

π cont iff $\forall U \in \mathcal{T}'$ on S
$\pi^{-1}(U) \in \mathcal{T}_X$
so $\mathcal{T}' \subseteq \mathcal{T}_S$ ← largest w/ this property

universal property of quotient topology:

- $\forall (Z, \mathcal{T}_Z)$ top sp.
and function of sets $f : S \to Z$ f is continuous iff
$f \circ \pi : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$ is continuous

$$\begin{array}{ccc} X & \longrightarrow & Z \\ & {}_{\pi}\searrow \quad \nearrow_{f} & \\ & S & \end{array}$$