Topology [3]

X set $\quad \mathcal{T}$ a topology on X

$\mathcal{T} = \{U_\alpha\}_{\alpha \in I}$ a collection of subsets of X (the "open sets" of X)

1) arbitrary unions of elements of $\mathcal{T}$ are in $\mathcal{T}$

$$U = \bigcup_{\alpha'} U_{\alpha'} \Rightarrow U \in \mathcal{T}$$

2) finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$

$$I = \bigcap_{i=1}^{N} U_{\alpha_i} \Rightarrow I \in \mathcal{T}$$

3) $\emptyset$ and X are in $\mathcal{T}$

$\mathcal{T}$ is closed under arbitrary unions and finite intersections and contains full space & empty set

$(X, \mathcal{T})$ a topological space

$x \in X$ point
$U_\alpha \in \mathcal{T}$ open sets

- a neighborhood of a pt $x \in X$ is any $U \in \mathcal{T}$ open set s.t. $x \in U$

- $\mathcal{T} = \{\emptyset, X\}$ is a topology — indiscrete topology $\mathcal{I}$
- $\mathcal{T} = P(X)$ all subsets are open is a topology — discrete topology $\mathcal{D}$

$X = \{0,1\}$

$\mathcal{T}_1 = P(X) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ discrete topology

$\mathcal{T}_2 = \{\emptyset, \{0\}, \{0,1\}\}$ Sierpinski topology

- $\mathbb{R} = X$ $\quad$ $\mathcal{U} \subset \mathbb{R}$ open if $\quad$ is a topology $\mathcal{T}$

$\forall x \in \mathcal{U} : \exists\, r > 0$ s.t. $B_r(x) \subset \mathcal{U}$

$\qquad\qquad B_r(x) = \{y \in \mathbb{R} \mid |x-y| < r\}$ $\qquad$ Euclidean topology

here see why want asymmetric condition between union & intersection:

$$\bigcup_\alpha \mathcal{U}_\alpha \supset \mathcal{U}_\alpha \supset B_r(x) \quad \text{so } \bigcup_\alpha \mathcal{U}_\alpha \in \mathcal{T}$$

but only for finite: $\displaystyle \underset{\underset{x}{\Psi}}{\bigcap_{i=1}^{N} \mathcal{U}_{\alpha_i}} \supset \bigcap_{i=1}^{N} B_{r_i}(x) \supset B(x, \underbrace{\min\{r_1, \ldots, r_N\}}_{})$

for fin. many this $r = \min\{r_1, \ldots, r_N\} > 0$

- topology $\mathcal{T} \subseteq \mathcal{P}(X)$

So set of all possible topologies on X is partially ordered by inclusions

- $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if $\mathcal{U} \in \mathcal{T}_1 \Rightarrow \mathcal{U} \in \mathcal{T}_2$
- $\mathcal{I} \subseteq \mathcal{T} \subseteq \mathcal{D}$ $\quad$ alway indiscrete & discrete smaller & largest possible
- larger or "finer" topology

smaller or "coarser" topology

- basis of a topology $\quad \mathcal{B} \subset \mathcal{T}$

a subcollection of open sets of a topology

$\{B_\gamma\}$ $\qquad B_\gamma \in \mathcal{T}$

s.t. $\forall\, \mathcal{U}_\alpha \in \mathcal{T}$ $\quad \mathcal{U}_\alpha = \bigcup_{\gamma'} B_{\gamma'}$ $\qquad$ each open set is a union of el'ts of basis

- if $\mathcal{B} \subset \mathcal{T}$ a basis: elements $B \in \mathcal{B}$ suffice to generate the topology $\mathcal{T}$ by taking arbitrary unions

e.g. $\mathcal{D} = \mathcal{T}$ discrete topology then $\mathcal{B} = \{\{x\} \mid x \in X\}$ basis

equivalent properties: *

1) $\mathcal{B}$ basis for $\mathcal{T}$

2) $\forall\, U \in \mathcal{T} \quad \forall\, x \in U \quad \exists\, B \in \mathcal{B} : \; x \in B \subseteq U$

1) ⇒ 2): $x \in U$ since $U \in \mathcal{T}$ & $\mathcal{B}$ basis $\exists\, B_\alpha$
$\bigcup_\alpha B_\alpha = U$ so $x \in B_\alpha$ for at least one of them

2) ⇒ 1) for each $x \in U$ let $B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq U$

then $\bigcup_{x \in U} B_x = U$ so $\mathcal{B}$ is a basis

ways of constructing topologies on a set X:

given a family $\Sigma = \{A_\alpha\}_{\alpha \in I}$ of subsets $A_\alpha \subset X$ (with no particular properties)

$\exists!$ smallest topology $\mathcal{T}(\Sigma) \supset \Sigma$

obtained by taking

1) $\emptyset, X$ (if not already in Σ)

2) all A_α in Σ — finite intersections

4) all $\bigcup_{\alpha'} I_{\alpha'}$; 3) all finite inters. $\bigcap_{i=1}^{N} A_{\alpha_i} = I$

- $\mathcal{T}(\Sigma)$ also the intersection of all topologies containing Σ (at least $P(X) = \mathcal{D}$ so nonempty)

it is clear this should contain all sets 1), 2), 3), 4)
but 1), 2), 3), 4) already give a topology containing Σ so should be smalles
(union distributes over $\cap$)

- in this construction $I_{\alpha_1 \ldots \alpha_N} = A_{\alpha_1} \cap \cdots \cap A_{\alpha_n}$
fin inters. of elt's of Σ
$\{ I_{\alpha_1 \ldots \alpha_N} \} = \mathcal{B}$ is a basis of $\mathcal{T}(\Sigma)$

- a set of subsets Σ is a <u>sub-basis</u> for $\mathcal{T}$ if $\mathcal{T} = \mathcal{T}(\Sigma)$
<u>i.e.</u> if fin. intersections of elt's of Σ are a basis for $\mathcal{T}$

- To determine topologies on X $\rightsquigarrow$ construct basis $\mathcal{B}$

systems of neighborhoods of <u>pts $x \in X$</u>

$\mathcal{N} = \{ \mathcal{U}_\alpha(x) \}_{\substack{\alpha \in I \\ x \in X}}$ if $y \in \mathcal{U}_\alpha(x) \cap \mathcal{U}_\beta(x')$
then $\exists\, \mathcal{U}_\gamma(y) \subset \mathcal{U}_\alpha(x) \cap \mathcal{U}_\beta(x')$

then $\{ \emptyset, X, \mathcal{U}_\alpha(x) \}_{\substack{\alpha \in I \\ x \in X}} = \mathcal{B}$ basis for a topology

form topology $\mathcal{T}(\mathcal{N})$ for which $\mathcal{N}$ is sub-basis
check in fact $\mathcal{N}$ is basis i.e. every fin inters. of
$\mathcal{U}_{\alpha_1}(x_1) \cap \cdots \cap \mathcal{U}_{\alpha_N}(x_N)$ is a union of $\mathcal{U}_\alpha(x)$: $\forall x \in \mathcal{U}$ $x \in \mathcal{U}_{\alpha'}(x') \subseteq \mathcal{U}$ some α', x'

Know true if $N=2$ by ; suppose true for $N-1$ $\quad x \in \mathcal{U}_{\alpha'}(x') \cap \mathcal{U}_{\alpha_N}(x_N)$
$\mathcal{U}_{\alpha_1}(x_1) \cap \cdots \cap \mathcal{U}_{\alpha_{N-1}}(x_{N-1}) = \mathcal{U} = \bigcup_{\alpha, x} \mathcal{U}_\alpha(x)$ $\quad \mathcal{U} \cap \mathcal{U}_{\alpha_N}(x_N) \ni x$ $\quad \Downarrow$ $\quad x \in \mathcal{U}_{\beta}(x) \subseteq$

so $\mathcal{U} \cap \mathcal{U}_{\alpha_N}(x_N) = \bigcup_{x \in \mathcal{U} \cap \mathcal{U}_{\alpha_N}(x_N)} \mathcal{U}_\beta(x)$

two bases $\mathcal{B}, \mathcal{B}'$ in X <u>equivalent</u> if
generate same topology $\mathcal{T} = \mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$

this happens iff both satisfied:

- $\forall B \in \mathcal{B}$ $\quad \forall x \in B$ $\quad \exists B' \in \mathcal{B}'$ $\quad x \in B' \subseteq B$
- $\forall B' \in \mathcal{B}'$ $\quad \forall x' \in B'$ $\quad \exists B \in \mathcal{B}$ $\quad x' \in B \subseteq B'$

if only one (e.g. first) then $\underset{\text{coarser}}{\mathcal{T}(\mathcal{B})} \subseteq \underset{\text{finer}}{\mathcal{T}(\mathcal{B}')}$

$\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$ implies both because of ~~def~~ basis (equiv. property ⊛)
other direction if first property holds for all $B \in \mathcal{B}$ then hold for all $U \in \mathcal{T}(\mathcal{B})$
so $\mathcal{B}'$ generate ~~by unions of~~ $\mathcal{T}(\mathcal{B})$ (hence $\mathcal{T}(\mathcal{B}')$ finer than $\mathcal{T}(\mathcal{B})$)
other direction from second: $\mathcal{B}$ generate by ~~unions of~~ $\mathcal{T}(\mathcal{B}')$
(hence $\mathcal{T}(\mathcal{B})$ finer than $\mathcal{T}(\mathcal{B}')$)

<u>Closed sets</u>: $C \subset X$ closed iff complement $C^c = U$ open

<u>e.g.</u> in $\mathcal{I} = \{\emptyset, X\}$ both $\emptyset, X$ open and closed

- in $\mathcal{D} = \mathcal{P}(X)$ all subsets of X open and closed
- in Euclidean $\mathcal{T}$ on $\mathbb{R}$ $\quad [a,b]$, $(-\infty, a]$, $[b, \infty)$
  examples of 'closed sets'
- in $\{0,1\}$ w/ $\{\emptyset, \{0\}, \{0,1\}\}$
  $\{1\}$ closed ~~and~~ and not open
  $\{0\}$ open and not closed
  $\emptyset, \{0,1\}$ open & closed

Closed sets family $\mathcal{C} \subset \mathcal{P}(X)$
closed under finite unions & arbitrary intersections:

$\left(\bigcup_{i=1}^{N} C_{\alpha_i}\right)^c = \bigcap_{i=1}^{N} C_{\alpha_i}^c$ open

$\left(\bigcap_{\alpha} C_\alpha\right)^c = \bigcup_{\alpha} C_\alpha^c$ open