Products and Coproducts in categories

(they don't always exist, it depends on the category)

Category of Sets $\quad X, Y \in \mathrm{Obj}(\mathrm{Sets})$

prod. $X \times Y$ is characterized by a univ. property

$\exists$ maps $\pi_X : X \times Y \to X \quad \pi_Y : X \times Y \to Y$ such that
for all maps $\quad f_1 : Z \to X \quad f_2 : Z \to Y \qquad Z \in \mathrm{Obj}(\mathrm{Sets})$

$\exists!\, f : Z \to X \times Y$ such that

diagr. commutes

$$\begin{tikzcd} & Z \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

$\pi_X \circ f = f_1$
$\pi_Y \circ f = f_2$

Same over arbitrary collection $\prod_{\alpha \in I} A_\alpha$ &

$\{f_\alpha : Z \to A_\alpha\}$

$\exists!\, f : \pi_\alpha \circ f = f_\alpha$

i.e. $f_1 : Z \to X \quad f_2 : Z \to Y$
can be assembled into a single map $f : Z \to X \times Y$
that recovers f_1, f_2 via projection

Coprod in Sets $\qquad$ disjoint union $\quad X \sqcup Y$

univ. property:

$\exists$ morphisms $\quad j_X : X \hookrightarrow X \sqcup Y \quad j_Y : Y \hookrightarrow X \sqcup Y$

s.t. for any $f_1 : X \to Z \quad f_2 : Y \to Z$

$\exists!\, f : X \sqcup Y \to Z$

$$\begin{tikzcd} X \arrow[r, hook, "j_X"] \arrow[dr, "f_1"'] & X \sqcup Y \arrow[d, "f"] & Y \arrow[l, hook', "j_Y"'] \arrow[dl, "f_2"] \\ & Z & \end{tikzcd}$$

$f \circ j_X = f_1$
$f \circ j_Y = f_2$

same for $\bigsqcup_\alpha A_\alpha$

$\{f_\alpha : A_\alpha \to Z\}$

$\exists!\, f : f \circ j_\alpha = f_\alpha$

diagrams in categories

e.g.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z \arrow[ur, "h"'] & \end{tikzcd}$$

commutative iff $f = h \circ g$

$X, Y, Z \in Obj(\mathcal{C})$
$f, g, h \in Mor_{\mathcal{C}}$

indexing category

3 objects

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \arrow[ur] & \end{tikzcd}$$

D

morphisms : 3 morphisms shown + $id_\bullet$ morphisms
+ all possible compositions

a diagram in $\mathcal{C}$ with this shape is a functor

$$F : D \longrightarrow \mathcal{C}$$

D-shaped diagram in $\mathcal{C}$ = functor $F \in Func(D, \mathcal{C})$

morphism of diagrams w/ same shape = nat. transf. of functors

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Cones and Co-cones

- constant diagram X is functor $D \to \mathcal{C}$ ($X \in Obj(\mathcal{C})$)
  assigning $\cancel{X}(j) = X \quad \forall j \in Obj(D)$ and $\cancel{X}(\phi) = 1_X$
  $\forall \phi \in Mor_D$

- cone over $F: D \to \mathcal{C}$ with tip X nat transf $\alpha: X \to F$

$$\begin{tikzcd} & X \arrow[dl, "\alpha_j"'] \arrow[dr, "\alpha_{j'}"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(j) \arrow[rr, "F(\phi)"'] & {} & F(j') \end{tikzcd}$$

$\forall \phi : j \to j'$ in D

- cocone under F tip X
  nat transf $\beta: F \to X$

$$\begin{tikzcd} F(j) \arrow[rr, "F(\phi)"] \arrow[dr, "\beta_j"'] & {} \arrow[d, phantom, "\circlearrowleft"] & F(j') \arrow[dl, "\beta_{j'}"] \\ & X & \end{tikzcd}$$

$\boxed{11B}$

Example: diagram
$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D \\ & C \arrow[ur] & \end{tikzcd}$$

cone:
$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[ddr, bend left] \arrow[dr] & \\ A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] & D \\ & C \arrow[ur] & \end{tikzcd}$$

cocone:
$$\begin{tikzcd} & B \arrow[r] \arrow[dd] & D \arrow[ddl] \\ A \arrow[ur] \arrow[r] \arrow[dr, bend right] & C \arrow[ur] \arrow[d] & \\ & X & \end{tikzcd}$$

where all triangles of arrow w/ X at one vertex are commutative (but original diagram may be or not)

- diagram $F: \mathcal{D} \to \mathcal{C}$
- Functor $\mathrm{Cone}(-,F): \mathcal{C}^{op} \longrightarrow \mathrm{Sets}$

$X \in \mathrm{Obj}(\mathcal{C}) \longmapsto \mathrm{Cone}(X,F)$ = the set of cones over F with tip F = the set of nat. transf $\alpha: X \to F$

$\underbrace{f: Y \to X}_{\in \mathrm{Mor}_{\mathcal{C}^{op}}(X,Y)} \in \mathrm{Mor}_{\mathcal{C}}(Y,X) \longmapsto \mathrm{Cone}(X,F) \to \mathrm{Cone}(Y,F)$

$\alpha \longmapsto \alpha\circ f$ $\quad (\alpha\circ f)_j = \alpha_j \circ f$

in example:
$$\begin{tikzcd} & Y \arrow[d] & \\ & X \arrow[dl] \arrow[d] \arrow[ddl, bend left] \arrow[dr] & \\ A \arrow[r] \arrow[d] & C \arrow[dr] & D \\ B \arrow[urr] & & \end{tikzcd}$$

- Functor $\mathrm{Cocone}(F,-): \mathcal{C} \to \mathrm{Sets}$

$X \longmapsto \mathrm{Cocone}(F,X)$ = set of all cones under F w/ tip X = set of all nat transf. $\beta: F \to X$

$f: X \to Y \in \mathrm{Mor}_{\mathcal{C}}(X,Y) \longmapsto \mathrm{Cocone}(F,X) \to \mathrm{Cocone}(F,Y)$

$\beta \longmapsto f\circ\beta$ $\quad (f\circ\beta)_j = f\circ\beta_j$

in example:
$$\begin{tikzcd} A \arrow[r] \arrow[dr] \arrow[ddr, bend right] & C \arrow[r] \arrow[dd] & D \arrow[ddl] \\ & B \arrow[ur] \arrow[d] & \\ & X \arrow[d] & \\ & Y & \end{tikzcd}$$

$\boxed{12B}$

a <u>limit</u> of diagram F (if it exists)

is an object in $Obj(\mathcal{C}) \ni \lim_D F$

that <u>represents</u> (Yoneda) $Cone(-, F) : \mathcal{C}^{op} \to Sets$

<u>i.e.</u> $Mor_{\mathcal{C}}(X, \lim_D F) \cong Cone(X, F)$

a <u>colimit</u> of diagr F (if it exists)

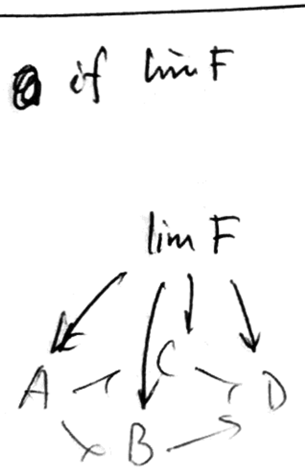
is an object $colim_D F \in Obj(\mathcal{C})$ that represents

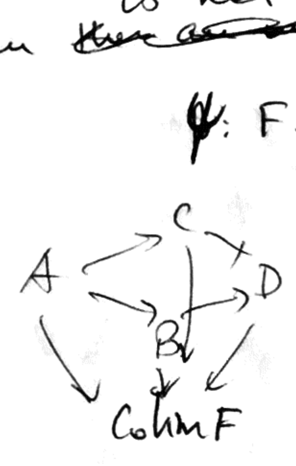
$Cocone(F, -)$ so that $Mor_{\mathcal{C}}(colim_D F, X) \cong Cocone(F, X)$

What this means:

if $\lim F$ $colim F$ exists then ~~there are morphisms~~ is not ~~most~~

$\phi : \lim F \to F$ $\psi : F \to colim F$





Such that given any other cone (tip X)

$\exists!$ ~~...~~ $u : X \to \lim_D F$ s.t.

$$\begin{tikzcd} X \arrow[d, dashed, "u"'] \arrow[dr, "\alpha_j"] & \\ \lim F \arrow[r, "\phi_j"'] & F(j) \end{tikzcd}$$

commutes

for any other cocone (tip X)

$\exists!$ $u : colim_D F \to X$

s.t.

$$\begin{tikzcd} F(j) \arrow[r, "\psi_j"] \arrow[dr, "\beta_j"'] & colim_D F \arrow[d, "u"] \\ & X \end{tikzcd}$$

commutes

lim & colim express the concept of <u>optimization</u> $\boxed{13B}$

they are the "optimal" cone (all other cones have to factor through)

or "optimal" cocone (all other cocones have to factor through)

> Most constructions in category theory are based on this notion of optimization

index cat	diagr	limit	colimit
		terminal obj	initial obj
$\bullet \quad \bullet \quad \bullet$	$A \quad B \quad C$	prod $A \times B \times C$	coprod $A \sqcup B \sqcup C$
$\bullet \to \bullet \leftarrow \bullet$	$A \to C \leftarrow B$	pullback $\begin{matrix} A \times_C B & \to & B \\ \downarrow & & \downarrow \\ A & \to & C \end{matrix}$	
$\bullet \leftarrow \bullet \to \bullet$	$A \leftarrow C \to B$		pushforward/pushout $\begin{matrix} C & \to & B \\ \downarrow & & \downarrow \\ A & \to & A *_C B \end{matrix}$
$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$	$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$	inverse lim $\varprojlim_i A_i$	
$\bullet \to \bullet \to \bullet \to \cdots$	$A_1 \to A_2 \to A_3 \to \cdots$		direct limit $\varinjlim_i A_i$
$\bullet \rightrightarrows \bullet$	$A \rightrightarrows B$	equalizer $E \to A \rightrightarrows B$	coequalizer $A \rightrightarrows B \to CoE$

initial & terminal objects

$\mathcal{D}$ empty $F : \mathcal{D} \to \mathcal{C}$ empty diagram

What is lim / colim

$T = \lim F \in Obj(\mathcal{C})$ s.t. $\forall X \in Obj(\mathcal{C})$ $\exists!\ u : X \to T$
"terminal object"

$I = \operatorname{colim} F \in Obj(\mathcal{C})$ st. $\forall X \in Obj(\mathcal{C})$ $\exists!\ u : I \to X$
"initial object"

an object that is both initial & terminal is a "zero-object"

Sets initial object $I = \emptyset$
terminal object $T = \{*\}$ one-point set

(if initial / terminal obj. exists then unique up to unique isomorphism)

$\mathcal{D}$ has objects & only identity morphisms

$F : \mathcal{D} \to \mathcal{C}$ a collection of objects $F(j) \in Obj(\mathcal{C})$ parametrized by $j \in Obj(\mathcal{D})$

$\lim_{\mathcal{D}} F$ is $\lim_{\mathcal{D}} F = \prod_{j} F(j)$ product in $\mathcal{C}$

product

as given cone $X \xrightarrow{f_j} F(j)$ $\exists!\ f : X \to \lim_{\mathcal{D}} F$
$\phi_j \circ f = f_j$
$\phi_j : \lim_{\mathcal{D}} F \to F(j)$

$\operatorname{colim}_{\mathcal{D}} F = \coprod_{j} F(j)$

coproduct given cocone $F(j) \xrightarrow{f_j} X$ $\exists!\ f : \operatorname{colim}_{\mathcal{D}} F \to X$
$f \circ \phi_j = f_j$ $\phi_j : F(j) \to \operatorname{colim}_{\mathcal{D}} F$

$\boxed{15B}$

$$D \quad \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{F} \begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$F : D \to \mathcal{C}$ diagram

$\lim_D F = X \times_Z Y$ pullback

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

s.t. for all cone C

$$\begin{tikzcd} C \arrow[dr, "u"] \arrow[drr, bend left, "u_X"] \arrow[ddr, bend right, "u_Y"'] & & \\ & X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

$\exists! \, u : C \to X \times_Z Y$

(for sets pullback is

$X \times_Z Y = \{ (x,y) \in X \times Y \mid g(y) = f(x) \in Z \}$)

$\operatorname{colim}_D F = X \sqcup_Z Y$

pushout
(or pushforward)

$$D \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \xrightarrow{F} \begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "a_X"] \\ Y \arrow[r, "a_Y"'] & X \sqcup_Z Y \end{tikzcd}$$

s.t. for all cocones

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "a_X"] \arrow[ddr, bend left, "u_X"] & \\ Y \arrow[r, "a_Y"] \arrow[drr, bend right, "u_Y"'] & X \sqcup_Z Y \arrow[dr] & \\ & & C \end{tikzcd}$$

$\exists! \, u : X \sqcup_Z Y \to C$

s.t. all digr. commute

for sets pushout is quotient of $X \sqcup Y$ by rel. gen by $f(z) \sim g(z)$

$\boxed{\text{if } f, g \text{ inclusions and } Z = X \cap Y \text{ then colim} = X \cup Y}$

equalizer & coequalizer in Sets;

$$\underset{D}{\bullet \rightrightarrows \bullet} \quad \xrightarrow{F} \quad X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

$E = \{ x \in X \mid f(x) = g(x) \}$

$CoE = Y/\sim$ eq. gen by

$f(x) \sim g(x)$

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inverse limit

$$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$$

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \leftarrow \cdots \leftarrow X_n \xleftarrow{f_n} \cdots$$

in Sets

$$\varprojlim_n X_n = \{ (x_n)_n \in \prod_n X_n \mid f_n(x_{n+1}) = x_n \}$$

direct limit

$$\bullet \to \bullet \to \bullet \to \bullet \to \cdots$$

$$X_1 \xrightarrow{f_1} X_2 \to X_3 \to \cdots \xrightarrow{f_n} X_n \to \cdots$$

or more gen. $X_i \xrightarrow{f_{ij}} X_j$, $i \le j$ w/ order on Obj(D)

$$\varinjlim_n X_n = \coprod_n X_n \Big/ \sim$$

$x_i \sim x_j$ if $\exists$ some k s.t. $i \le k$, $j \le k$ and $f_{ik}(x_i) = f_{jk}(x_j)$

$x_{n-1} \sim x_n$ if

here if $f_n(x_{n-1}) = x_n$