(1B)

# Category Theory

$\mathcal{C}$ category: (1) $Obj(\mathcal{C})$ class of objects (set: small category)

(2) $\forall C_1, C_2 \in Obj(\mathcal{C})$ $Mor_{\mathcal{C}}(C_1, C_2)$ set of morphisms ((locally) small category)

(Notation: $\mathcal{C}(C_1, C_2)$ or $Hom_{\mathcal{C}}(C_1, C_2)$)

$f: C_1 \to C_2 \in Mor_{\mathcal{C}}(C_1, C_2)$ [but not nec. functions]

(3) Composition rule:

if $f \in Mor_{\mathcal{C}}(C_1, C_2)$ & $g \in Mor_{\mathcal{C}}(C_2, C_3)$

$\exists$ $g \circ f \in Mor_{\mathcal{C}}(C_1, C_3)$

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

Conditions:

(i) associativity of composition

$f \in Mor_{\mathcal{C}}(C_1, C_2)$, $g \in Mor_{\mathcal{C}}(C_2, C_3)$, $h \in Mor_{\mathcal{C}}(C_3, C_4)$

$h \circ (g \circ f) = (h \circ g) \circ f$

(ii) existence of identity morphisms

$\forall C \in Obj(\mathcal{C})$ $\exists$ $1_C \in Mor_{\mathcal{C}}(C, C)$

s.t. $1_C \circ f = f$ $\forall f \in Mor_{\mathcal{C}}(C, D)$

$f \circ 1_C = f$

$1_C$ must be unique as $1_C' \circ 1_C = 1_C$

$1_C''$

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Example: **Sets** Obj(Sets) are sets

$\mathrm{Mor}_{Sets}(X,Y) =$ functions $f: X \to Y$

(but not small category)

**$Sets_*$** $Obj(Sets_*)$ pointed sets $(X, x_0)$

$\mathrm{Mor}_{Sets_*}(X,Y) = \{ f: X \to Y : f(x_0) = y_0 \}$

**FinSets** finite sets (small category)

**$FinSets_*$** finite pointed sets (also)

**Grps** obj = groups Mor = group homomorphisms

**$Vect_K$** obj = (fin dim) vector spaces over field $K$

mor = linear maps

**Top** → obj topological spaces

mor = continuous functions

— partially ordered set $(S, \leq)$ category $\mathcal{C}_{(S,\leq)}$ $\begin{cases} Obj = s \in S \\ \mathrm{Mor}_{\mathcal{C}}(s,s') = \begin{cases} \text{single mor if } s \leq s' \\ \emptyset \text{ otherwise} \end{cases} \end{cases}$

reflexive, transitive, asymmetric ($s \leq s'$ & $s' \leq s \Rightarrow s = s'$)

— $G$ directed graph (or multigraph: allowed multiple & looping edges)

↳ Category $\mathcal{C}_G$ $Obj(\mathcal{C}_G) = V_G$ vertices of $G$

$\mathrm{Mor}_{\mathcal{C}_G}(v,v') =$ directed paths in $G$ w/ source $v$ and target $v'$

— Opposite category $\mathcal{C}^{opp}$ reversing all arrows

$Obj(\mathcal{C}^{opp}) = Obj(\mathcal{C})$ $\mathrm{Mor}_{\mathcal{C}^{opp}}(X,Y) \equiv \mathrm{Mor}_{\mathcal{C}}(Y,X)$

$f: X \to Y$ in $\mathrm{Mor}_{\mathcal{C}}(X,Y)$

pullback by $f$

$$f^*: \mathrm{Mor}_{\mathcal{C}}(Y,Z) \to \mathrm{Mor}_{\mathcal{C}}(X,Z)$$

$$\begin{tikzcd} Y \arrow[d, "g"'] & X \arrow[l, "f"'] \arrow[dl, dashed, "g\circ f"] \\ Z & \end{tikzcd}$$

$f^*(g) = g \circ f$ precomposition by $f$

pushforward by $f$

$$f_*: \mathrm{Mor}_{\mathcal{C}}(Z,X) \to \mathrm{Mor}_{\mathcal{C}}(Z,Y)$$

$$\begin{tikzcd} Z \arrow[d, "h"'] \arrow[dr, "f\circ h"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$f_*(h) = f \circ h$ post composition by $f$

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$f: X \to Y$ isomorphism ~~iff~~ equivalent to

(1) $\forall Z \in \mathrm{Obj}(\mathcal{C})$ $f_*: \mathrm{Mor}_{\mathcal{C}}(Z,X) \xrightarrow{\simeq} \mathrm{Mor}_{\mathcal{C}}(Z,Y)$ isom of sets

(2) $\forall Z \in \mathrm{Obj}(\mathcal{C})$ $f^*: \mathrm{Mor}_{\mathcal{C}}(Y,Z) \xrightarrow{\simeq} \mathrm{Mor}_{\mathcal{C}}(X,Z)$ isom of sets

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show (1) equiv to $f: X \to Y$ isom

if $f: X \to Y$ isom then

$\exists\, g: Y \to X$ inverse

then $\forall Z$ $\quad g_*$ is inverse of $f_*$

$f_*(h) = f \circ h \quad g_*(f_*(h)) = g \circ (f \circ h) = (g \circ f) \circ h = h$

$f_*(g_*(h)) = h$ same way

if $f_*: \mathrm{Mor}_{\mathcal{C}}(Z, X) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(Z, Y)$ isom of sets $\forall Z$

pick $Z = Y$ then $f_*: \mathrm{Mor}_{\mathcal{C}}(Y, X) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(Y, Y)$

$\Rightarrow f: X \to Y$ has to be surjective

as have $1_Y = f \circ g$ for some $g: Y \to X$

$f_*: \mathrm{Mor}_{\mathcal{C}}(X, X) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(X, Y)$ for $Z = X$

$f_*(1_X) = f$

$f_*(g \circ f) = f \circ g \circ f = 1_Y \circ f = f = f_*(1_X)$

but $f_*$ inj so $g \circ f = 1_X$ so $f$ inj.

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Functors $\quad F: \mathcal{C} \to \mathcal{C}'$ $\left\{ \begin{array}{l} F: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}') \\ F: \mathrm{Mor}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Mor}_{\mathcal{C}'}(F(C_1), F(C_2)) \end{array} \right.$ (covariant)

compatible w/ composition and units!

$$\begin{cases} F(g) \circ F(f) = F(g \circ f) \\ F(1_C) = 1_{F(C)} \end{cases}$$

contravariant functor

$F: \mathcal{C}^{op} \to \mathcal{C}'$

reverses direction of arrows

Functor $F : \mathcal{C} \to \mathcal{C}'$

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y))$$
$$f \longmapsto F(f)$$

1) $F$ faithful if this map injective $f \mapsto F(f)$
2) $F$ full if " " surjective "
3) $F$ fully faithful if " " both (bijective) "

Note: functors describe invariants of ~~[illegible]~~ objects in $\mathcal{C}$ up to isomorphisms

because a functor (is compat. w/ unit & composition) maps isomorphisms to isomorphism

Important use in topology
topological invariants ~~[illegible]~~
functors from category Top to other categories (groups, rings, algebras, etc.)

Examples of functors
- forgetful functor: Grps $\longrightarrow$ Sets
  views grp $G$ as underlying set (forgets grp str)
  grp homom as map of sets
- $\mathcal{C}$ small category $C \in \mathrm{Obj}(\mathcal{C})$ functor $\mathrm{Mor}_{\mathcal{C}}(C,-) : \mathcal{C} \to$ Sets
  $$X \longmapsto \mathrm{Mor}_{\mathcal{C}}(C,X)$$
  $$f : X \to Y \longmapsto f_* : \mathrm{Mor}_{\mathcal{C}}(C,X) \to \mathrm{Mor}_{\mathcal{C}}(C,Y)$$

## Natural transformations

Comparing functors $F, G : \mathcal{C} \to \mathcal{C}'$

two functors in what way related or "same" ?

$\eta : F \to G$ natural transformation

consists of

$\forall X \in Obj(X)$ a morphism $\eta_X \in Mor_{\mathcal{C}'}(F(X), G(X))$

$\forall f : X \to Y \in Mor_{\mathcal{C}}(X, Y)$ commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

- invertible natural transf. (all $\eta_X$ invertible)
  = isomorphisms of functors

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Example directed graph $G$ $V_G$, $E_G$ oriented edges
so source, target $s(e), t(e) \in V_G$

- Category $\mathcal{2}$ has 2 objects $V, E$
  and only 2 non-identity morphisms
  $s, t : E \rightrightarrows V$ (no compositions possible except w/ identity)
- $G$ is a functor $G : \mathcal{2} \to Sets$

$G(E) = E_G$ set of edges $\quad G(s) = s : E_G \to V_G$
$G(V) = V_G$ set of vertices $\quad G(t) = t : E_G \to V_G$ source & target maps

$G' \subset G$ subgraph

$G' : 2 \to Sets \quad G : 2 \to Sets$ functors

$\exists\ \eta : G' \to G$ natural transformation

i.e. $\eta_V : V_{G'} \to V_G$ functions between sets $\forall$ vertices and edges

$\eta_E : E_{G'} \to E_G$

that compatible w/ s, t maps

$$\begin{tikzcd} E_{G'} \arrow[r, "\eta_E"] \arrow[d, "t,\ s"'] \arrow[dr, phantom, "\circlearrowright"] & E_G \arrow[d, "s,t"] \\ V_{G'} \arrow[r, "\eta_V"'] & V_G \end{tikzcd}$$

subgraph then means $\eta_E, \eta_V$ are inclusions

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## Yoneda lemma

for $F, G$ functors $Nat(F,G) = \{$all natural transf. $\eta : F \to G\}$

$\mathcal{C}$ category and $C \in Obj(\mathcal{C})$

$F : \mathcal{C}^{op} \to Sets$ a contravariant functor

consider also functor $Mor_{\mathcal{C}}(-, C)$ (also contravariant, $: \mathcal{C}^{op} \to Sets$)

$$Nat(Mor_{\mathcal{C}}(-,C), F) \cong F(C) \quad \text{isomorphic as sets}$$

in particular if

$F = Mor_{\mathcal{C}}(-,D)$ then $Nat(Mor_{\mathcal{C}}(-,C), Mor_{\mathcal{C}}(-,D)) \cong Mor_{\mathcal{C}}(C,D)$

(objects can be studied through their morphisms from/to)

Notation

motivated by set-theoretic
notation for Cartesian products

$$X^n = \text{Functions}(f : \{1, \ldots, n\} \to X)$$

$\mathcal{C}$ category, $\mathcal{D}$ category

$\mathcal{D}^{\mathcal{C}} := \text{Func}(\mathcal{C}, \mathcal{D})$ category of functors

objects are functors from $\mathcal{C}$ to $\mathcal{D}$
morphisms are natural transformations

(see example of directed graphs)