

Category Theory

(set, small catg)

(1B)

\mathcal{C} category: (1) $\text{Obj}(\mathcal{C})$ class of objects

(locally) small category

(2) $\forall C_1, C_2 \in \text{Obj}(\mathcal{C})$ $\text{Mor}_{\mathcal{C}}(C_1, C_2)$ set of morphisms

(Notation: $\text{Ecc}_{\mathcal{C}}(C_1, C_2)$ or $\text{Hom}_{\mathcal{C}}(C_1, C_2)$)

$f: C_1 \rightarrow C_2 \in \text{Mor}_{\mathcal{C}}(C_1, C_2)$ [but not nec. functions]

(3) Composition rule:

if $f \in \text{Mor}_{\mathcal{C}}(C_1, C_2)$ & $g \in \text{Mor}_{\mathcal{C}}(C_2, C_3)$

\exists $g \circ f \in \text{Mor}_{\mathcal{C}}(C_1, C_3)$

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

Conditions:

(i) associativity of composition

$f \in \text{Mor}_{\mathcal{C}}(C_1, C_2)$, $g \in \text{Mor}_{\mathcal{C}}(C_2, C_3)$, $h \in \text{Mor}_{\mathcal{C}}(C_3, C_4)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(ii) existence of identity morphisms

$\forall C \in \text{Obj}(\mathcal{C}) \exists 1_C \in \text{Mor}_{\mathcal{C}}(C, C)$

s.t. $1_C \circ f = f \quad \forall f \in \text{Mor}_{\mathcal{C}}(C, D)$

$$f \circ 1_C = f$$

1_C must be unique as $\underset{1_C}{1_C'} \circ 1_C = 1_C$

Example: Sets Obj(Sets) are sets
Mor_{Sets}(X, Y) = functions $f: X \rightarrow Y$
(but not small category)

Sets* Obj(Sets*) pointed sets (X, x_0)
Mor_{Sets*}(X, Y) = $\{ f: X \rightarrow Y : f(x_0) = y_0 \}$

Fin Sets finite sets (small category)

Fin Sets* finite pointed sets (also)

Grps Obj = groups Mor = group homomorphisms

Vect_K Obj = (finite) vector spaces over field K
Mor = linear maps

Top Obj topological spaces
Mor = continuous functions

reflexive, transitive, asymmetric
 $(S \leq S' \wedge S' \leq S \Rightarrow S = S')$

— partially ordered set (S, \leq) category $\mathcal{C}_{(S, \leq)}$ $\left\{ \begin{array}{l} \text{Obj} = \{ S \in S \} \\ \text{Mor}_{\mathcal{C}}(S, S') = \begin{cases} \text{single map if } S \leq S' \\ \emptyset \text{ otherwise} \end{cases} \end{array} \right.$

— G directed graph (or multigraph: allowed multiple & looping edges)

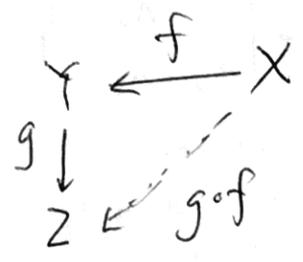
↳ Category \mathcal{C}_G Obj(\mathcal{C}_G) = V_G vertices of G
Mor _{\mathcal{C}_G} (v, v') = directed paths in G w/ source v and target v'

— opp into category \mathcal{C}^{opp} reversing all arrows
Obj(\mathcal{C}^{opp}) = Obj(\mathcal{C})
Mor _{\mathcal{C}^{opp}} (X, Y) = Mor _{\mathcal{C}} (Y, X)

$$f: X \rightarrow Y \quad \text{in } \text{Mor}_{\mathcal{C}}(X, Y)$$

pullback by f

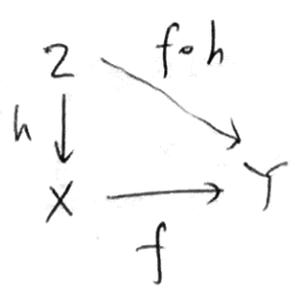
$$f^*: \text{Mor}_{\mathcal{C}}(Y, Z) \rightarrow \text{Mor}_{\mathcal{C}}(X, Z)$$



$$f^*(g) = g \circ f \quad \text{precomposition by } f$$

pushforward by f

$$f_*: \text{Mor}_{\mathcal{C}}(Z, X) \rightarrow \text{Mor}_{\mathcal{C}}(Z, Y)$$



$$f_*(h) = f \circ h \quad \text{post composition by } f$$

$f: X \rightarrow Y$ isomorphism ~~iff~~ equivalent to

(1) $\forall Z \in \text{Obj}(\mathcal{C}) \quad f_*: \text{Mor}_{\mathcal{C}}(Z, X) \xrightarrow{\cong} \text{Mor}_{\mathcal{C}}(Z, Y)$ isom of sets

(2) $\forall Z \in \text{Obj}(\mathcal{C}) \quad f^*: \text{Mor}_{\mathcal{C}}(Y, Z) \xrightarrow{\cong} \text{Mor}_{\mathcal{C}}(X, Z)$ isom of sets

show (1) equiv to $f: X \rightarrow Y$ isom

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if $f: X \rightarrow Y$ isom then

$\exists g: Y \rightarrow X$ inverse
then $\exists g_*$ is inverse of f_*

$$f_*(h) = f \circ h \quad g_*(f_*(h)) = g \circ (f \circ h) = (g \circ f) \circ h = h$$

$$f_*(g_*(h)) = h \text{ same way}$$

if $f_*: \text{Mor}_e(Z, X) \xrightarrow{\cong} \text{Mor}_e(Z, Y)$ isom of sets $\forall Z$

pick $Z = Y$ then $f_*: \text{Mor}_e(Y, X) \xrightarrow{\cong} \text{Mor}_e(Y, Y)$

$\Rightarrow f: X \rightarrow Y$ has to be surjective

as have $1_Y = f \circ g$ for some $g: Y \rightarrow X$

~~pick $Z = X$ then $f_*: \text{Mor}_e(X, X) \xrightarrow{\cong} \text{Mor}_e(X, Y)$~~

~~$f_*(1_X) = f$~~

$f_*: \text{Mor}_e(X, X) \xrightarrow{\cong} \text{Mor}_e(X, Y)$

for $Z = X$ ~~$f_*(1_X) = f$~~

$$f_*(g \circ f) = f \circ g \circ f = 1_Y \circ f = f = f_*(1_X)$$

but f_* inj ~~so~~ so $g \circ f = 1_X$ so f inj.

Functors $F: \mathcal{C} \rightarrow \mathcal{C}'$ $\left\{ \begin{array}{l} F: \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C}') \text{ (covariant)} \\ F: \text{Mor}_{\mathcal{C}}(C_1, C_2) \rightarrow \text{Mor}_{\mathcal{C}'}(F(C_1), F(C_2)) \end{array} \right.$

compatible w/ composition and units:

$$\begin{cases} F(g) \circ F(f) = F(g \circ f) \\ F(1_C) = 1_{F(C)} \end{cases}$$

contravariant functors

$F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}'$
reverses direction of arrows

Functor $F: \mathcal{C} \rightarrow \mathcal{C}'$

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$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}'}(F(X), F(Y))$

$f \mapsto F(f)$

- 1) F faithful if this map injective $f \mapsto F(f)$
- 2) F full if " " surjective " "
- 3) F fully faithful if " " both (bijective) " "

Note: functors describe invariants of ~~objects~~ objects in \mathcal{C} up to isomorphisms

because a functor (is ^{is} comput. w/ unit & comparison)
maps isomorphisms to isomorphisms

Important use in topology

topological invariants ~~invariants~~

functors from category Top to other categories
(groups, rings, algebras, etc.)

Examples of functors

- forgetful functor: $\text{Grps} \rightarrow \text{Sets}$
views grp G as underlying set (forgets grp str)
grp homom as map of sets

- \mathcal{C} ^{small} category $\mathcal{C} \in \text{Ob}(\mathcal{C})$ functor $\text{Mor}_{\mathcal{C}}(\mathcal{C}, -): \mathcal{C} \rightarrow \text{Sets}$
 $X \mapsto \text{Mor}_{\mathcal{C}}(\mathcal{C}, X)$

$f: X \rightarrow Y \mapsto f_*: \text{Mor}_{\mathcal{C}}(\mathcal{C}, X) \rightarrow \text{Mor}_{\mathcal{C}}(\mathcal{C}, Y)$

Natural transformations

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Comparing functors

$$F, G: \mathcal{C} \rightarrow \mathcal{C}'$$

two functors in what way related or "same" ?

$\eta: F \rightarrow G$ natural transformation
consists of

$\forall X \in \text{Obj}(\mathcal{C})$ a morphism $\eta_X \in \text{Mor}_{\mathcal{C}'}(F(X), G(X))$

$\forall f: X \rightarrow Y \in \text{Mor}_{\mathcal{C}}(X, Y)$ commutative diagram

$$\begin{array}{ccccc}
 X & & F(X) & \xrightarrow{\eta_X} & G(X) \\
 f \downarrow & & F(f) \downarrow & \circlearrowleft & \downarrow G(f) \\
 Y & & F(Y) & \xrightarrow{\eta_Y} & G(Y)
 \end{array}$$

- invertible natural transf. (all η_X invertible)
= isomorphisms of functors

Example directed graph G V_G, E_G oriented edges
so source, target $s(e), t(e) \in V_G$

- Category \mathcal{I} has 2 objects V, E
and only 2 non-identity morphisms

$$s, t: E \rightrightarrows V \quad (\text{no compositions possible except w/ identity})$$

- G is a functor $G: \mathcal{I} \rightarrow \text{Sets}$

$$\begin{array}{ll}
 G(E) = E_G \text{ set of edges} & G(s) = s: E_G \rightarrow V_G \\
 G(V) = V_G \text{ set of vertices} & G(t) = t: E_G \rightarrow V_G
 \end{array}$$

source
target

$G' \subset G$ subgraph

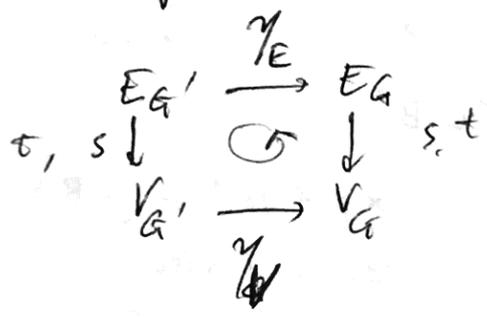
$G': \mathcal{Z} \rightarrow \text{Sets}$ $G: \mathcal{Z} \rightarrow \text{Sets}$ functors

$\exists \eta: G' \rightarrow G$ natural transformation

we $\eta_V: V_{G'} \rightarrow V_G$ functions between sets of vertices and edges

$\eta_E: E_{G'} \rightarrow E_G$

that compatible w/ s,t maps



subgraph then means η_E, η_V are inclusions

Yoneda lemma

for F, G functors $\text{Nat}(F, G) = \{ \text{all natural transf. } \eta: F \rightarrow G \}$

\mathcal{C} category and $c \in \text{Obj}(\mathcal{C})$

$F: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ a contravariant functor

consider also functor $\text{Mor}_{\mathcal{C}}(-, c)$ (also contravariant) $: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$

$\text{Nat}(\text{Mor}_{\mathcal{C}}(-, c), F) \cong F(c)$ isomorphic as sets

in particular if

$F = \text{Mor}_{\mathcal{C}}(-, D)$ then $\text{Nat}(\text{Mor}_{\mathcal{C}}(-, c), \text{Mor}_{\mathcal{C}}(-, D)) \cong \text{Mor}_{\mathcal{C}}(c, D)$
(objects can be studied through their morphisms from/to)

Notation

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motivated by set-theoretic
notation for Cartesian products

$$X^n = \text{Functions}(f: \{1, \dots, n\} \rightarrow X)$$

\mathcal{C} category, \mathcal{D} category

$\mathcal{D}^{\mathcal{C}} := \text{Func}(\mathcal{C}, \mathcal{D})$ category of functors

objects are functors from \mathcal{C} to \mathcal{D}
morphisms are natural transformations

(see example of directed graphs)
