X path connected, locally path connected, semilocally simply connected

$\forall$ $H \subseteq \pi_1(X, x_0)$ subgroup of the fundamental group

$\exists$ a path connected covering space $\tilde{X}_H$

$$\begin{array}{c} \tilde{X}_H \\ \downarrow \pi_H \\ X \end{array}$$

s.t. $\pi_{H*} : \pi_1(\tilde{X}_H, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$

under image $\pi_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

Pf: $\tilde{X}$ univ. cover (simply connected covering space)

$[\alpha], [\alpha'] \in \tilde{X}$ equivalence relation $[\alpha] \sim_H [\alpha']$

iff $\alpha(1) = \alpha'(1)$ and $[\underbrace{\alpha \cdot \alpha'^{-1}}_{\text{loop}}] \in H$

(it is an equivalence relation)

Take $\tilde{X}_H = \tilde{X} / \sim_H$ quotient top. space w/ resp. to this equivalence relation

note: in the basis $\mathcal{U}_{[\alpha]}$ of topology of $\tilde{X}$

if a pt $[\beta] \in \mathcal{U}_{[\alpha]}$ identified by $\sim_H$ w/

a pt. $[\beta'] \in \mathcal{U}_{[\alpha']}$ then all of $\mathcal{U}_{[\alpha]}$ identified w/ all of $\mathcal{U}_{[\alpha']}$

($[\alpha \cdot \beta]$ β path in $\mathcal{U}$ $\beta(0) = \alpha(1)$) $\alpha\beta(1) = \alpha'\beta'(1)$ & $[\alpha\beta \cdot \beta'^{-1}\alpha'^{-1}] \in H$ $\Rightarrow$ then also $[\alpha \cdot \beta \cdot \gamma] \sim_H [\alpha' \cdot \beta' \cdot \gamma]$

- so π_H restr. of π to identified copies of $U_{[\alpha]}$ still a covering map
- path connected (image of $\tilde{X}$ under quotient map by $\sim_H$)
- $\tilde{x}_0 \in \tilde{X}_H$ class of $[c_{x_0}]$ under $\sim_H$ equiv.

$\qquad\qquad\qquad = \tilde{x}_0$ in $\tilde{X}$

loop $\gamma \in \pi_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_0))$ loop at x_0 whose lift to $\tilde{X}_H$ is a loop

path in $\tilde{X}_H$ $\quad t \mapsto [\alpha_t]$ loop if $[\alpha] \sim_H [c_{x_0}]$

i.e. if $\alpha(1) = x_0$ and $[\alpha \cdot c_{x_0}^{-1}] = [\alpha] \in H$

i.e. $\pi_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

X path conn., loc. path conn., w/ path conn. covering spaces

$\exists$ isom. of cov. spaces

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "\pi_2"] \\
& (X, x_0) &
\end{tikzcd}$$

iff $\pi_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = \pi_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ coincide as subgroups in $\pi_1(X, x_0)$

Pf: given isom. ϕ

$\pi_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))$ subgp of $\pi_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ and vice versa using

$\pi_1 = \pi_2 \circ \phi \qquad \pi_2 = \pi_1 \circ \phi^{-1}$

$\Rightarrow$ if ~~loop in~~ γ loop in $\tilde{X}_1$ based at $\tilde{x}_1$ $\Rightarrow$ $\pi_1 \circ \gamma$ loop in X based at x_0

$\qquad = \pi_2 \circ \phi \circ \gamma \Rightarrow \phi \circ \gamma$ loop in $\tilde{X}_2$ based at $\tilde{x}_2$

if $\pi_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = \pi_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ ③

then $\exists$ lift $\tilde{\pi}_1 : \tilde{X}_1 \to \tilde{X}_2$ of π_1 (by lifting criterion)

and choose uniquely so that $\tilde{\pi}_1(\tilde{x}_1) = \tilde{x}_2$

Similarly $\exists$ lift $\tilde{\pi}_2 : \tilde{X}_2 \to \tilde{X}_1$ of π_2 ~~xxxx~~ $\tilde{\pi}_2(\tilde{x}_2) = \tilde{x}_1$

$\Rightarrow$ $\pi_1 = \pi_1 \circ (\tilde{\pi}_2 \circ \tilde{\pi}_1)$ fixing basept $\tilde{x}_1$

lifting of π_1 (w/ resp. to π_1 itself!)

uniqueness of such lift (w/ assigned base pt.)

$\Rightarrow$ $id_{\tilde{X}_1}$ also a lift $\Rightarrow$ $\tilde{\pi}_2 \circ \tilde{\pi}_1 = id_{\tilde{X}_1}$

Same for reverse composition

so $\phi = \tilde{\pi}_2 \circ \tilde{\pi}_1$ is isomorphism of covering spaces

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$$\left\{ \begin{array}{c} \text{subgroups} \\ H \\ \text{of } \pi_1(X, x_0) \end{array} \right\} \xleftrightarrow{\text{one-to-one corresp.}} \left\{ \begin{array}{c} \text{isom. classes} \\ \text{of covering} \\ \text{spaces} \\ \tilde{X}_H \xrightarrow{\pi_H} X \end{array} \right\}$$

Seifert - van Kampen theorem

- free product of groups $\{G_\alpha\}_{\alpha\in J}$ family of groups

$$G = \mathop{\ast}_{\alpha\in J} G_\alpha \qquad \text{(sometimes written as } \coprod^{\ast}_{\alpha\in J} G_\alpha \text{)}$$

if all G_α subgroups of G with
~~[illegible]~~ $G_\alpha \cap G_\beta = \{e\}$ only id. when $\alpha \neq \beta$
and all elements of G have a unique reduced word representation in terms of elements of G_α

i.e. $g = g_{\alpha_1} \cdots g_{\alpha_n} \qquad g_{\alpha_i} \in G_{\alpha_i}$

where no successive aa^{-1} elements occur (in any g_{α_i})
and $g_{\alpha_i} \neq e$

- main property of free product

for any group H and homomorphisms $h_\alpha : G_\alpha \to H$

$\exists!$ homom. $h : \mathop{\ast}_\alpha G_\alpha \longrightarrow H$ s.t. $h|_{G_\alpha} = h_\alpha$

- amalgamated product of groups (push forward in the category of groups)

G_1, G_2 groups $\quad N_1, N_2$ normal subgroups $\quad N_1 \triangleleft G_1$, $N_2 \triangleleft G_2$
N smallest normal subgroup of $G_1 * G$
that contains both N_1 & N_1

then $G/N \cong (G_1/N_1) * (G_2/N_2)$ (compatibility of free prods w/ quotient grps)

- suppose groups & group homomorphisms

N = norm. subgr. of $G_1 * G_2$ gen. by elements $\phi_1(h)\phi_2(h)^{-1}$ $h \in H$ (smallest norm. subgr. containing these)

$$\begin{tikzcd} H \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & G_1 \arrow[d, "\Phi_1"] \\ G_2 \arrow[r, "\Phi_2"] & G_1 *_H G_2 := (G_1 * G_2)/N \end{tikzcd}$$

$\exists \tilde{\Phi}_2$ homom.

- main property of pushforward of groups

if have morphism $f_1 : G_1 \to G'$ $\quad f_2 : G_2 \to G'$ such that

$$f_1 \circ \phi_1 (h) = f_2 \circ \phi_2 (h) \quad \forall h \in H$$

then $\exists!$ map $f : G_1 *_H G_2 \to G'$ such that diagram commutes

$$\begin{tikzcd}
H \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & G_2 \arrow[d, "\Phi_2"] \arrow[ddr, bend left, "f_2"] & \\
G_1 \arrow[r, "\Phi_1"'] \arrow[drr, bend right, "f_1"'] & G_1 *_H G_2 \arrow[dr, "f"] & \\
 & & G'
\end{tikzcd}$$

<u>Note</u> $G_1 *_H G_2$ is <u>uniquely</u> characterized by this "universal property"

- Key algebraic tool for computing fundamental groups by decomposing spaces into simpler pieces

$X = U \cup V$ $\qquad$ want to compute $\pi_1(X)$ from $\pi_1(U)$, $\pi_1(V)$, $\pi_1(U \cap V)$ $\quad$ assuming these are easier to compute

assume U and V are open and path connected and $U \cap V$ is path connected

then there is a pushforward of groups

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "i_{U*}"] \arrow[d, "i_{V*}"'] & \pi_1(U) \arrow[d] \arrow[ddr, bend left, "j_{U*}"] & \\
\pi_1(V) \arrow[r] \arrow[drr, bend right, "j_{V*}"'] & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \arrow[dr, "\cong"] & \\
 & & \pi_1(X)
\end{tikzcd}$$

i_{U*}, i_{V*} morphisms induced by inclusions

$i_U : U \cap V \hookrightarrow U$

$i_V : U \cap V \hookrightarrow V$

j_{U*} and j_{V*} hom. induced by inclusions

$j_U : U \hookrightarrow U \cup V = X$

$j_V : V \hookrightarrow U \cup V = X$

the main claim of Seifert van Kampen theorem: this map is isomorphism

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

there is a boring proof (decompose loop in X into loops in U & V etc.) and a nicer proof that uses covering spaces

for these we also assume all $U, V, U \cap V, X$ are semilocally simply connected so we can use existence of univ. coverings

in order to use universal property of $G_1 *_H G_2$ need to know how to describe a homomorphism

$\pi_1(X) \to G'$ in terms of covering spaces

$$\mathrm{Hom}_{\mathrm{Grps}}(\pi_1(X), G') \underset{\text{bijection}}{\longleftrightarrow} \left\{ \begin{matrix} \tilde{X} \\ \downarrow \\ X \end{matrix} \; G'\text{-covering spaces} \right\}$$

- first observe: gluing of spaces and gluing of coverings

$$\begin{matrix} \tilde{U} & & \tilde{V} \\ \pi_U \downarrow & & \downarrow \pi_V \\ U & & V \end{matrix}$$

covering spaces and assume

$$\begin{matrix} \tilde{U} \supset \pi_U^{-1}(U \cap V) & \underset{\text{isom of cov-sp's}}{\cong} & \pi_V^{-1}(U \cap V) \subset \tilde{V} \\ \downarrow \pi_U & & \downarrow \pi_V \\ U \cap V & & U \cap V \end{matrix}$$

then $\exists$ $\begin{matrix} \tilde{X} \\ \pi \downarrow \\ X \end{matrix}$ s.t.

$$\begin{matrix} \pi^{-1}(U) & \underset{\text{isom of cov. sp.}}{\cong} & \tilde{U} \\ \pi \downarrow & & \downarrow \pi_U \\ U & & U \end{matrix} \quad \text{and} \quad \begin{matrix} \pi^{-1}(V) & \underset{\text{isom cov sp's}}{\cong} & \tilde{V} \\ \downarrow \pi & & \downarrow \pi_V \\ V & & V \end{matrix}$$

- Now use this to prove Seifert van Kampen

now have $f_1 : \pi_1(U) \to G'$ $f_2 : \pi_1(V) \to G'$ that agree on inters: $\pi_1(U \cap V)$)

want to find $f : \pi_1(X) \to G'$ $\qquad f_1 \circ i_{U*} = f_2 \circ i_{V*}$

s.t. $f \circ j_{U*} = f \circ j_{V*}$

start w/ univ. cover $\begin{matrix} \tilde{U}_{\text{univ.}} \\ \downarrow \\ U \end{matrix}$ fiber $\cong \pi_1(U)$ (because subgr. it corr. to is trivial grp.)

construct new covering space $\begin{matrix} \tilde{U}_{G'} \\ \downarrow \\ U \end{matrix}$ where fibers are copies of G' by taking quotient of $G' \times \tilde{U}_{\text{univ}}$ by equiv rel.

$x \in \tilde{U}$, $g \in G$, $g_1 \in \pi_1(U)$ $\quad (g f_1(g_1), x) \sim (g, g_1 \cdot x)$ — action of $\pi_1(U)$ on fiber of $\tilde{U}$

fiber over x is equiv. class $[g,x]$ $g \in G$ (still a cov. space) but possibly disconnected (not a case of classif thm) ⑦

same thing over V: start w/ universal cover $\tilde{V}_{univ} \to Y$ and obtain a cov. w/ fiber $\cong G$

- they agree when restricted to $U \cap V$ because univ covering spaces agree and f_1 & f_2 agree so equiv. classes same

- Now glue together these G fibered cov. spaces so obtain one over X

since this covering space $\tilde{X}_G \to X$ must have action of $\pi_1(X)$ on fiber hence $f: \pi_1(X) \to G$ homomorphism

and this must agree w/ previous $f_1(g_1)$ or $f_2(g_2)$ when restricted to $g = j_{u*}(g_1)$ $g = j_{v*}(g_2)$

- Now need to show this $f: \pi_1(X) \to G$ is uniquely determined by this construction ← i.e. that $\pi_1(X)$ generated by images of $\pi_1(U)$ & $\pi_1(V)$

look at univ. cover $\tilde{X}_{univ} \to X$

if only generate smaller subgroup $H \subset \pi_1(X)$ then would see restriction of $\tilde{X}_{univ}$ to U & V action $\pi_1(U)$ & $\pi_1(V)$ preserve this H

⇒ can obtain new sub-covers of $\tilde{X}_{univ}/U$ and $\tilde{X}_{univ}/V$ w/ fiber $= H$ that can be glued to $\tilde{X}_H \hookrightarrow \tilde{X}$ subcover over X

but $\pi_1(\tilde{X}) = 1$ simply conn

subcover of a covering

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\phi} & \tilde{Y} \\ \pi \downarrow & \swarrow \pi|_{\tilde{Y}} & \\ X & & \end{array}$$

$\phi_*: \pi_1(\tilde{Y}) \to \pi_1(\tilde{X})$

$\pi_*(\pi_1(\tilde{X})) = H \subset \pi_1(X)$

$(\pi|_{\tilde{Y}})_*(\pi_1(\tilde{Y})) = \pi_* \circ \phi_*(\pi_1(\tilde{Y}))$
$\subset H$ subgr.

but if $\tilde{X} = \tilde{X}_{univ}$. $\pi_1(\tilde{X}_{univ}) = 1$

so $H = 1$

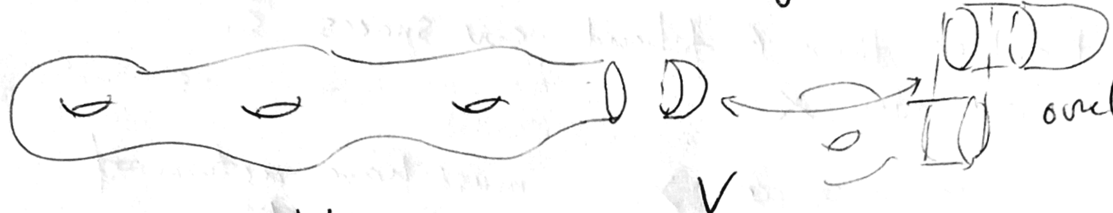
so $\pi_1(\tilde{Y}) = 1$ also because $(\pi|_{\tilde{Y}})_*$ inj. and image $\subset H$

Example:

$$\pi_1(\Sigma_g)$$

$\Sigma_g = \mathbb{T}^2 \underbrace{\# \cdots \#}_{g\text{-times}} \mathbb{T}^2$ topological surface

cut out a small disk from Σ_g w/ neck collars



so that

$U \cap V \simeq S^1 \times I$

covering spaces: univ. cover $\mathbb{R}$
fiber $\simeq \mathbb{Z}$

$\pi_1(U \cap V) = \mathbb{Z} = \pi_1(S^1)$

$\pi_1(V) = 1$ disk is contractible to pt
all loops are htpy trivial

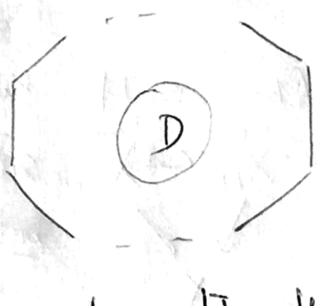
so

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \pi_1(\Sigma_g \setminus D) \\ {\scriptstyle \text{trivial map}}\downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(\Sigma_g) \end{array}$$

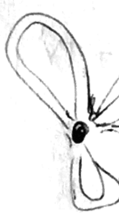
$\pi_1($  $) = \mathbb{Z} \underbrace{* \cdots *}_{g} \mathbb{Z}$

free product
again repeatedly use SvK
one circle at a time glued at pt.
$((\mathbb{Z} * \mathbb{Z}) * \mathbb{Z}) \cdots * \mathbb{Z}$

Use polygonal presentation
to see what is
$\pi_1(\Sigma_g \setminus D)$



open up the hole
until left w/ small band
around boundary

$\Rightarrow$  g loops like this

in quotient all vertices
identified