- ## Arbitrary Cartesian products of sets

revisit product of two sets $A \times B = \{(a,b) \mid a \in A \ \& \ b \in B\}$

$= \{$ functions $f : \{1,2\} \to A \cup B$ s.t. $f(1) \in A$ and $f(2) \in B \}$

$\{A_\alpha\}_{\alpha \in I}$ family of sets define Cartesian prod $\prod_{\alpha \in I} A_\alpha$

$$\prod_{\alpha \in I} A_\alpha = \{ f : I \to \bigcup_{\alpha \in I} A_\alpha \mid \forall \alpha \in I \ f(\alpha) \in A_\alpha \}$$

$x \in \prod_{\alpha \in I} A_\alpha$ $\quad x = (x_\alpha)_{\alpha \in I}$ $\quad x_\alpha = f(\alpha)$ w/ notation above

<u>Note</u>: if index set I finite clear that $\prod_{\alpha \in I} A_\alpha \neq \emptyset$ all $A_\alpha \neq \emptyset$
if set I is infinite? (axiom of choice)

- ## More set theory basics

finite set A $=$ $\exists$ bijection $f : A \to \{1, \dots, n\}$ for some $n \in \mathbb{N}$

$n = \# A$ **cardinality of A**

well defined i.e. property of A i.e. cannot have bijections
$A \xrightarrow{\simeq} \{1, \dots, n\}$ and $A \xrightarrow{\simeq} \{1, \dots, m\}$ w/ $n \neq m$

otherwise would have bijection
$\{1, \dots, n\} \xrightarrow[\simeq]{h} \{1, \dots, m\}$ injective & surjective

injective $\Rightarrow m \geq n$ $\quad h(1), \dots, h(n)$ n different pts in $\{1, \dots, m\}$

surjective $\Rightarrow m \leq n$ $\quad$ for each $j \in \{1, \dots, m\}$ $\exists i \in \{1, \dots, n\}$ $f(i) = j$

② in particular if $B \subsetneq A$ proper subset and ~~there~~ A finite nonempty there is **not** bijection $f: B \xrightarrow{\sim} A$

1) $\exists f: A \xrightarrow{\sim} \{1, \dots, n\}$ bijection

$\exists a \in A$ $a \notin B$ since $B \subsetneq A$

$f(a) \in \{1, \dots, n\}$ $\quad f(B) \subseteq \{1, \dots, n\} \setminus \{f(a)\}$

$\exists$ bijection $\{1, \dots, n\} \setminus \{f(a)\} \xrightarrow{\sim} \{1, \dots, n-1\}$
(permutation)

$\Rightarrow \# f(B) \underset{f \text{ bijection}}{=} \# B \leq n-1$

Infinite sets

a set is finite $\Rightarrow$ no bijection to a proper subset

$\exists$ bijection of finite sets $A \xrightarrow{\sim} B$ iff same cardinality $\# A = \# B$

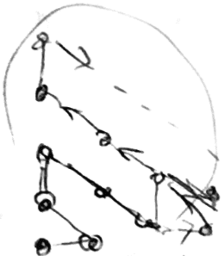
- infinite sets have bijections to proper subsets

  e.g. $2\mathbb{N} \hookrightarrow \mathbb{N}$ map $f(2k) = k$ is bijection (injective & surjective)

- A countably infinite (countable) iff

  $\exists$ bijection $f: A \to \mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$

  e.g. $\mathbb{Z}_+ \times \mathbb{Z}_+$ countable through diag. enumeration:



also through $\mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$
$f(n,m) = 2^n 3^m$
injective

- there are ~~countable~~ infinite sets that are not countable (uncountable)

[discuss below]

- countable unions of countable sets are countable $\{A_\alpha\}_{\alpha \in J}$

  $f_\alpha : \mathbb{Z}_+ \to A_\alpha$

  $h: \mathbb{Z}_+ \to J$

  $\mathbb{Z}_+ \times \mathbb{Z}_+ \xrightarrow{\text{surjective}} \bigcup_{\alpha \in J} A_\alpha$

  $(n, m) \longmapsto f_{h(n)}(m)$

in fact sufficient to see $\exists$ surjection $\mathbb{Z}_+ \twoheadrightarrow A$
to see A countable : to get bijection choose a preimage for each $a \in A$

get bijection to ~~subset of~~ (infinite) subset of $\mathbb{Z}_+$ (countable)

subset of countable set countable or finite

Note: equivalent facts
- A at most countable (bijection to finite set or $\mathbb{Z}_+$)
- $\exists$ injection $A \hookrightarrow \mathbb{Z}_+$
- $\exists$ surjection $\mathbb{Z}_+ \twoheadrightarrow A$

$f : \mathbb{Z}_+ \underset{\text{surj.}}{\twoheadrightarrow} A \Rightarrow g : A \underset{\text{injection}}{\rightarrow} \mathbb{Z}_+$ $\quad g(a) =$ smallest element n in $\mathbb{Z}_+$ with $f(n) = a$

$g : A \hookrightarrow \mathbb{Z}_+$ inj $\Rightarrow$ g bijection to a subset $g(A) \subset \mathbb{Z}_+$

all subsets of $\mathbb{Z}_+$ either finite or countably infinite
count pts of C in order in which occur in $\mathbb{Z}_+$
(get enumeration of C)

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set of all functions

$$\{ f : \mathbb{N} \to \{0,1\} \} = \{0,1\}^{\mathbb{N}}$$

$f = (a_1, a_2, a_3, \ldots, a_n, \ldots) \quad a_i \in \{0,1\}$
all possible infinite sequences of 0's and 1's

the set $\{0,1\}^{\mathbb{N}}$ is <u>not</u> countable

show there is no surjection $g : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$

(4) $g(n) = (a_{n,1}, a_{n,2}, \dots, a_{n,m}, \dots)$

each digit in $\{0,1\}$

take $y = (y_1, y_2, \dots, y_n, \dots)$ element in $\{0,1\}^{\mathbb{N}}$
defined by

$$y_n = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0 \end{cases}$$

$y \notin g(\mathbb{N})$

for each map $g: \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ can construct such a $y \notin g(\mathbb{N})$ so no surjections

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Note: $\{0,1\}^{\mathbb{N}} \cong \mathcal{P}(\mathbb{N})$ set of all subsets of $\mathbb{N}$

General procedure for jumping "cardinality" of infinite sets X and $\mathcal{P}(X)$

given any map $g: X \to \mathcal{P}(X)$ g is <u>not surjective</u>

$\forall\, x \in X$ $g(x)$ is a subset of X (which may or not contain pt x)

define another subset of X by rule

$$B = \{x \in X \mid x \notin g(x)\}$$

$\Rightarrow g_{\,ac}$ not surjective thus $B \notin g(A)$

is $B \in g(A)$? if so there would be $a_0 \in A$

s.t. $B = g(a_0)$ but ~~by construction~~

no $B \notin g(A)$

if $a_0 \in B$ then $a_0 \in g(a_0)$ contradicts $x \in B$ iff $x \notin g(x)$
if $a_0 \notin B$ then $a_0 \notin g(a_0)$ contradicts $x \in B$ iff $x \notin g(x)$

the axiom of choice and infinite sets

- A infinite set $\Rightarrow \exists\ f : \cancel{\mathbb{N}} \hookrightarrow A$ injection

  choose $a_1 \in A$ set $f(1) = a_1$
  suppose already defined $f(k)$ $\forall k \in \{1, \ldots, n-1\}$
  $A \setminus f(\{1, \ldots, n-1\})$ nonempty (because A infinite)
  choose $a_n \in A \setminus f\{1, \ldots, n-1\}$ and set $f(n) = a_n$

  inductively construct injective map $f : \mathbb{N} \hookrightarrow A$

- A infinite set $\Rightarrow \exists$ bijection A to proper subset of A

  use previous $f : \cancel{\mathbb{N}} \hookrightarrow A$ $\qquad B = f(\mathbb{Z}_+) \subseteq A$

  $f(n) = a_n \in A \qquad a_n \neq a_m$ for $n \neq m$ (injective)

  define $g : A \to A \setminus \{a_1\}$ by $\begin{cases} g(a_n) = a_{n+1} & a_n \in B \\ g(x) = x & x \notin B \end{cases}$

  is a bijection

<u>Note</u> used choices of a_n's to construct $f : \mathbb{Z}_+ \hookrightarrow A$

<u>Axiom of choice</u>: given an arbitrary (finite or infinite) collection $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of disjoint nonempty sets

$\exists$ a set C that consists of exactly one element from each A_α

$\exists$ a choice function $c : \mathcal{A} \longrightarrow \bigcup_\alpha \mathcal{A}_\alpha \qquad C = \{c(A_\alpha)\}_{\alpha \in I}$

s.t. $c(A_\alpha) \in A_\alpha$

6 Order relations and well ordered sets

- binary relation $R \subset X \times X$ is a preorder if
  - reflexive $\forall x \; (x,x) \in R$
  - $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$ transitive

(unlike equivalence not symmetric)

$$x \overset{R}{\prec} y \quad \text{if} \quad (x,y) \in R \quad \text{preorder}$$

- (an equiv. el. a particular case of preorder that also symmetric)
- on $\mathbb{R}$ $\quad x \leq y$ is preorder
- $\mathcal{P}(X)$ with inclusion $A \subseteq B$ of subsets is preorder

- partial ordering is a preordering w/ additional condition

$$\left(x \overset{R}{\prec} y \right) \wedge \left(y \overset{R}{\prec} x \right) \Rightarrow x = y$$

(i.e. only case w/ symm property holds is if $x = y$)

- total ordering partial ordering when $\forall x, y \in X$ either $x \overset{R}{\prec} y$ or $y \overset{R}{\prec} x$

- partially ordered set X is well ordered if each $B \subset X$ $B \neq \emptyset$ has a first element $b_0 \in B$ s.t. $b_0 \overset{R}{\prec} b$ $\forall b \in B$

- a <u>well ordered set</u> X is <u>totally ordered</u>:

take $\{a,b\} \subset X$ subset $\exists$ min el. So either $a \overset{R}{\lesssim} b$ or $b \overset{R}{\lesssim} a$

<u>Equivalent statements</u>:
1) axiom of choice
2) Zorn's lemma
3) Zermelo's ~~well~~ ordering

<u>Zorn's lemma</u>:
X preordered set: if each subset $B \subseteq X$ totally ordered (chain) ~~has maximal element~~ has upper bound $\Rightarrow$ X has at least one maximal element

some x s.t. $b \underset{R}{\lesssim} x \quad \forall b \in B$

<u>Zermelo's well ordering</u>: every set admits a $\overset{R}{\lesssim}$ that is a well-ordering

This proof not covered in class: added here for completeness

<u>Outline of proof</u>: <u>preliminaries</u>:

1) φ-towers $\quad X$ set $\quad \mathcal{F} \subseteq \mathcal{P}(X)$ family
$\varphi: \mathcal{F} \to X$ function

- $\emptyset \in \mathcal{F}$
- $\{A_\alpha\}_{\alpha \in I}$ totally ordered (by inclusion) $\quad A_\alpha \in \mathcal{F}$
$\Rightarrow \bigcup_\alpha A_\alpha \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A \cup \{\varphi(A)\} \in \mathcal{F}$

- $\mathcal{M}$ minimal φ-tower (intersection of all still φ-tower)

2) $\mathcal{F}$ φ-tower: $\exists A \in \mathcal{F}$ st. $\varphi(A) \in A$

show $\mathcal{M}$ <u>totally ordered by inclusion</u>, then $A = \bigcup\{M : M \in \mathcal{M}\}$

$A \in \mathcal{M} \Rightarrow A \cup \{\varphi(A)\} \in \mathcal{M} \Rightarrow A \cup \{\varphi(A)\} \subseteq A \Rightarrow \varphi(A) \in A$

(8) then **Axiom choice ⇒ Zorn**:

- $\mathcal{F} \subset \mathcal{P}(X)$ set of all chains in X (totally ordered subsets)
- $A \in \mathcal{F}$, $A \neq \emptyset$, a_A an upper bound (assuming each chain has an upper bound want to show then has max)

$$T_A = \{x \in X : (a_A \prec x) \wedge \neg(x \prec a_A)\}$$

$\boxed{\text{* want to show } T_A = \emptyset}$

(because this means either no x with $a_A \prec x$ or if $a_A \prec x$ then also $x \prec a_A$ so a_A is maximal)

suppose $T_A \neq \emptyset$ and assuming axiom of choice so $\exists$ choice function c that chooses an element in each T_A $c(A) \in T_A$

and $c(\emptyset) = x_0$

— then show $\mathcal{F}' = \mathcal{F} \cup \{\emptyset\}$ is a c-tower

- $\emptyset \in \mathcal{F}'$
- $A_\alpha \in \mathcal{F}'$ s.t. $\{A_\alpha\}$ is a chain (by inclusion) then $\bigcup_\alpha A_\alpha \in \mathcal{F}'$: $a, b \in \bigcup_\alpha A_\alpha$ $a \in A_\alpha$ $b \in A_\beta$ $A_\alpha \subset A_\beta$ or viceversa

  so a, b related

  so $\bigcup_\alpha A_\alpha$ tot. ordered

- $A \in \mathcal{F}' \Rightarrow A \cup \{c(A)\} \in \mathcal{F}'$ enough to see $a \in A$ and $c(A)$ related

  $a \prec a_A$ and $\underbrace{a_A \prec c(A)}_{\text{in } T_A}$ $\Rightarrow a \prec c(A)$

— apply previous lemma: $\exists A \in \mathcal{F}'$: $c(A) \in A$ hence $c(A) \prec a_A$

but contradiction because $c(A) \in T_A \Rightarrow \neg(c(A) \prec a_A)$

$\Rightarrow T_A = \emptyset$

Well ordering ⇒ axiom of choice:

$\{A_\alpha\}$ family, take well ordering on $\bigcup_\alpha A_\alpha$, define choice function $c(A_\alpha)$ = first element in A_α w/ resp. to this well ordering

Zorn $\Rightarrow$ Well ordering

E set — some subsets can be well ordered
want to show E itself can also

- $\mathcal{F}$ family of all ~~ord~~ pairs $(A, \prec_A)$
  $A \subset E$ $\quad \prec_A$ is a well ordering on A

i.e. all subsets that can be well ordered

- put an ordering on $\mathcal{F}$

$(A, \prec_A) < (B, \prec_B)$ iff

$$\left.\begin{array}{l} * \ A \subset B \\ * \ \prec_B \text{ restricted to } A \text{ induces } \prec_A \\ * \ y \in B \setminus A \ \& \ x \in A \Rightarrow x \prec_B y \end{array}\right\} (*)$$

is a partial ordering $(\mathcal{F}, <)$

check partially ordered set $(\mathcal{F}, <)$ satisfies requirements of Zorn's lemma

$\{ (A_\alpha, \prec_\alpha) \}$ a chain in $(\mathcal{F}, <)$

$\Rightarrow \bigcup_\alpha A_\alpha$ has a well ordering $\prec_U$ (i.e. is in $\mathcal{F}$)

so that $(\bigcup_\alpha A_\alpha, \prec_U)$ is upper bound for

this works if define $\prec_U$ to be:

$a, b \in \bigcup_\alpha A_\alpha \rightsquigarrow$ there is ~~some~~ $A_\alpha \ni a, b$
(because chain ordered by inclusion)

then $a \prec_U b$ iff $a \prec_\alpha b$ in this. well defined because in a larger one would be same

so $\prec_U$ well def. partial ordering

$<_U$ is also well-ordering because of $*$

because $<_U$ satisfies $(y \in A_\alpha) \wedge (x <_U y) \Rightarrow x \in A_\alpha$

$\Rightarrow$ if $Q \subset \bigcup_\alpha A_\alpha$ $Q \neq \emptyset$ there is a first A_α w/ $Q \cap A_\alpha \neq \emptyset$
and first element in $Q \cap A_\alpha$ is also first element in Q

So $\underbrace{(\bigcup_\alpha A_\alpha, <_U)}_{}$ $\in \mathcal{F}$ and $(A_\alpha, <_\alpha) < (\bigcup_\alpha A_\alpha, <_U)$

so is an upper bound ~~and~~ in $\mathcal{F}$

Zorn $\Rightarrow$ $\mathcal{F}$ has a maximal element $(A, <_A)$
$(\mathcal{F}, <)$ this must have $A = E$

otherwise if $a_0 \in E \setminus A$

could get $(A, <_A) < (A \cup \{a_0\}, <')$
where in $<'$ a_0 is last element
and $<_A$ on A

and would not be maximal