

- "functoriality of fundamental group"

X, Y top. spaces $f: X \rightarrow Y$ contin. function

$x_0 \in X \quad f(x_0) = y_0 \in Y \quad \exists$ group homomorphism

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

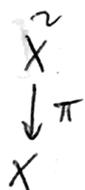
$$f_* (\gamma) (t) = f \circ \gamma (t) \quad \text{loop based at } f(x_0) \neq$$

if $\gamma: [0,1] \rightarrow X$ loop
 $\gamma(0) = \gamma(1) = x_0$

if $\gamma_1 \sim \gamma_2$ homotopic $h: I \times I \rightarrow X$ then composition compatible
 $f_* (\gamma_1) \sim f_* (\gamma_2)$ via $f \circ h$ and $f_1 \circ f_2 \rightsquigarrow (f_1)_* \circ (f_2)_*$

- if X path connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1) \quad \forall x_0, x_1 \in X$
 (write $\pi_1(X)$)

- Covering space



projection map (surj., contin.)
 $\forall x \in X$ st. $\exists U \subseteq X$ open $\pi^{-1}(U)$ is a disjoint union of open sets U_α

$$\pi^{-1}(U) = \bigsqcup_{\alpha} U_{\alpha} \quad \text{with} \quad \pi|_{U_{\alpha}}: U_{\alpha} \xrightarrow{\cong} U \quad \text{homeom.}$$

locally a ~~same~~ disj. union of copies of base
 but not globally

Example: $\pi: \mathbb{R} \rightarrow S^1$

$$\pi(x) = \exp(ix)$$

around each $\theta = e^{ix} \in S^1$

if δ small union disjoint:

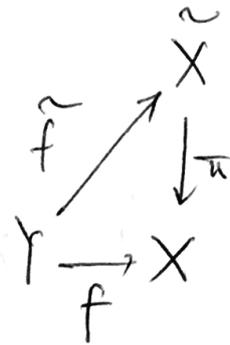
$$\pi^{-1}(I_{\delta}) = \bigsqcup_n (x - \delta + 2\pi n, x + \delta + 2\pi n)$$

\exists interval $(e^{i(x-\delta)}, e^{i(x+\delta)}) = I_{\delta}$

(2) • covering space \tilde{X} and contin. function
 $\downarrow \pi$
 X $f: Y \rightarrow X$

• lift of f = $\tilde{f}: Y \rightarrow \tilde{X}$ st.

$$f = \pi \circ \tilde{f}$$



• homotopy lifting property: \tilde{X} covering space
 $\downarrow \pi$
 X

$F: Y \times I \rightarrow X$ homotopy (between $F(y,0) = f_0(y)$ and $F(y,1) = f_1(y)$)

if f_0 admits a lift \tilde{f}_0 then $\exists!$ lift \tilde{F}

homotopy of \tilde{f}_0 to \tilde{f}_1 with \tilde{f}_t lift of f_t for all $t \in I$

Pf: $y_0 \in Y$ $U = U(y_0)$ open neighb.

know $\tilde{F}: U \times \{0\} \rightarrow \tilde{X}$ exists given by \tilde{f}_0

~~know~~ suppose have extended \tilde{F} to $U \times [0, \tau]$ for some $\tau > 0$

take $F(U \times [\tau, \tau'])$ for some $\tau' > \tau$ w/ $\tau' - \tau$ small so that

\exists open set $V \subset X$ on which π^{-1} is local homeom.

w/ $F(U \times [\tau, \tau']) \subset V$ ($\pi^{-1}(V) = \cup_{\alpha} V_{\alpha}$ $V_{\alpha} \cong V$)

$$F(y_0, \tau) \in V$$

$\tilde{F}(y_0, \tau) \in V_{\alpha}$ for one α

take $\pi_{\alpha} = \pi|_{V_{\alpha}}: V_{\alpha} \cong V$

~~then~~ then take $\tilde{F}(y, t) = \pi^{-1}(F(y, t)) \cap V_{\alpha}$
 $(y, t) \in U \times [\tau, \tau']$

choose this particular copy in the preimage

\rightarrow continue

• When $Y = \{pt\}$ path lifting property

$$f: I \rightarrow X \quad \text{once chosen } \tilde{x}_0 \in X \quad \pi(\tilde{x}_0) = x_0$$

$$f(0) = x_0 \quad \exists! \tilde{f}: I \rightarrow \tilde{X} \quad \pi \circ \tilde{f} = f$$

$$\tilde{f}(0) = \tilde{x}_0$$

• if $f: I \rightarrow X$ loop $f(0) = f(1) = x_0$ and $\tilde{x}_0 \in \tilde{X} \quad \pi(\tilde{x}_0) = x_0$

$\Rightarrow \tilde{f}: I \rightarrow \tilde{X}$ path $\tilde{f}(1) \in \pi^{-1}(x_0)$ but in general not same as $\tilde{f}(0) = \tilde{x}_0$

• $\tilde{X} \ni \tilde{x}_0 \quad w/ \pi(\tilde{x}_0) = x_0$

$\downarrow \pi$

$X \ni x_0$

induced morphism

$$\pi_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$$

is injective

image $\pi_* (\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$

$\{ \gamma \text{ loops } / \text{htpy} \text{ in } X \text{ based at } x_0 \text{ such that lift } \tilde{\gamma} \text{ starting at } \tilde{x}_0 \text{ is a loop} \}$

- to see that $\ker(\pi_*) = 1$: an element $[\gamma] \in \ker(\pi_*)$

γ loop w/ $\pi \circ \gamma \sim c_{x_0}$ constant loop at x_0

but htpy lifting property \Rightarrow htpy between γ and c_{x_0}

• existence of lifts: $(\tilde{X}, \tilde{x}_0) \xrightarrow{\pi} (X, x_0)$ condition for existence of lift

$(Y, y_0) \xrightarrow{f} (X, x_0)$ $f_* (\pi_1(Y, y_0)) \subseteq \pi_* (\pi_1(\tilde{X}, \tilde{x}_0))$ subgroup

if lift \tilde{f} have $\pi \circ \tilde{f} = f$ so $\pi_* \circ \tilde{f}_* = f_*$ so \uparrow OK

(4) if $f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\tilde{X}, \tilde{x}_0))$ (*)

$y \in Y$ γ path y_0 to y \rightsquigarrow path lifting:
 $f \circ \gamma$ path x_0 to $f(y)$ $\exists!$ $\tilde{f} \circ \gamma: I \rightarrow \tilde{X}$
 start at \tilde{x}_0 w/
 $\pi(\tilde{f} \circ \gamma(1)) = f(y)$

define $\tilde{f}(y) := \tilde{f} \circ \gamma(1)$

need to know well defined indep of γ used

if $\tilde{f} \circ \gamma(1)$ and $\tilde{f} \circ \gamma'(1)$ $\gamma \circ \gamma'$ loop in Y at y_0

$f_*([\gamma \circ \gamma'])$ loop in X at x_0 by (*)

$\exists \alpha$ in \tilde{X} at \tilde{x}_0 $\pi_*([\alpha]) = f_*([\gamma \circ \gamma'])$

ie. $\pi \circ \alpha \stackrel{\text{hpic}}{=} f \circ (\gamma \circ \gamma')$
 through h_t

lifting homotopies $\Rightarrow \exists!$ \tilde{h}_t between α and path obtained by
 by uniqueness $\tilde{h}_1 = (\tilde{f} \circ \gamma) \cdot (\tilde{f} \circ \gamma')^{-1}$

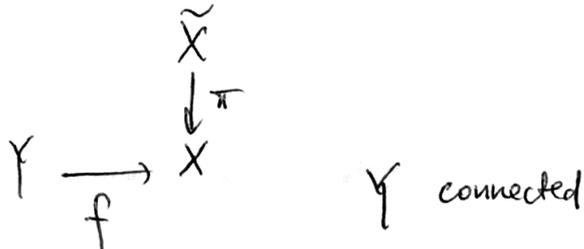
htpy of loops at \tilde{x}_0

$\rightsquigarrow (\tilde{f} \circ \gamma)(1) = (\tilde{f} \circ \gamma')(1)$

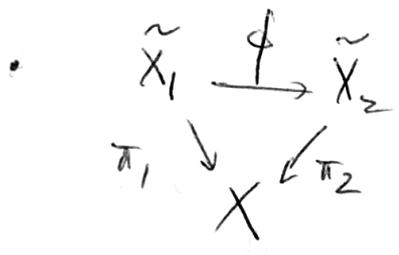
concatenation of these paths makes

\tilde{f} well defined

uniqueness of lifting ~~property~~ for paths extends to (similar pt)



if \tilde{f}_1, \tilde{f}_2 lifts of f
 agree at one pt
 $\Rightarrow \tilde{f}_1 = \tilde{f}_2$ everywhere



equivalence of covering spaces;
 homeomorphism ϕ
 compatible w/ projectors

$$\pi_2 \circ \phi = \pi_1$$

path connected with $\pi_1(X) = 1$

X semi-locally simply connected

$\forall x \in X \exists U(x)$ open neighb. s.t. $\pi_1(U(x), x) \xrightarrow{i_x^*} \pi_1(X, x)$
 $i: U(x) \hookrightarrow X$ inclusion is trivial

(Note $\pi_1(U(x), x)$ need not be trivial but for any (nontrivial) loop in $U(x) \exists$ htpy in X to constant loop)

if X is path connected, locally path connected, semi-locally simply connected

$\Rightarrow \exists$ a covering $\tilde{X} \xrightarrow{\pi} X$ w/ \tilde{X} simply connected

- Points of \tilde{X} are htpy classes of paths in X starting at x_0 (htpy relative to bdry)
 - $\pi: \tilde{X} \rightarrow X \quad \pi([\alpha]) = \alpha(1)$
 surjective because X path connected well defined

- topology on \tilde{X} :

$U = \{ U \subseteq X \text{ open} \mid \pi_1(U) \xrightarrow{i_x^*} \pi_1(X) \text{ trivial} \}$
 the local path conn + semi-local simply conn $\Rightarrow U$ basis for topology of X

$U \in U \quad \alpha$ path in X from x_0 to $x \in U$
 $U \cap \alpha I = \{ [\alpha \cdot \beta] \mid \beta \text{ path in } U; \beta(0) = \alpha(1) \}$ class of htpy rel. to ∂I

(6) $U_{[\alpha]} \subseteq \tilde{X}$ by construction

and $\tilde{U} = \{ U_{[\alpha]} \mid U \in \mathcal{U} \text{ a path} \}$
 is a basis for a topology on \tilde{X} (skip explicit checking)

$\pi: \tilde{X} \rightarrow X$ is covering map w/ resp. to these topologies

$\pi: U_{[\alpha]} \rightarrow U$ surjective (because U path conn.)

$x \in U \rightsquigarrow$ pick β path in U ~~with~~ $\beta(0) = \alpha(1)$ $\beta(1) = x$

so that $\alpha\beta \xrightarrow{\pi} x$; if β, β' paths same endpoints
 using $\pi_1(U, \alpha(1)) \xrightarrow{i_*} \pi_1(X, \alpha(1))$ trivial
 $\Rightarrow \beta \sim \beta'$

$\Rightarrow [\alpha, \beta] = [\alpha, \beta'] \Rightarrow \pi|_{U_{[\alpha]}}: U_{[\alpha]} \rightarrow U$ also injective

$\pi^{-1}(U) = \bigcup U_{[\alpha]}$
~~or~~ coincide $U_{[\alpha]} = U_{[\alpha']}$
 (or disjoint $U_{[\alpha]}$)

\tilde{X} simply connected; if is path connected

α path in X starting at x_0
 $\alpha_t(s) = \begin{cases} \alpha(s) & 0 \leq s \leq t \\ \alpha(t) & t \leq s \leq 1 \end{cases}$ \leftarrow like α on $[0, t]$
 \leftarrow constant $\alpha(t)$ on $[t, 1]$

$t \mapsto [\alpha_t]$ map is path from constant $[c_{x_0}]$ path to $[\alpha]$
 hence \tilde{X} path conn.

base pt. $\tilde{x}_0 \in \tilde{X}$
 \parallel
 $[c_{x_0}]$

Know $\pi_x: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$
 is injective

image of loop: $[\alpha]$ that lifts to $[\alpha_t]$ $\xrightarrow{\text{loop}}$ so check if $\pi_x(\pi_1(\tilde{X}, \tilde{x}_0))$ is trivial
 $\alpha_1 = \alpha$ must be $\alpha_1 \sim c_{x_0} \Rightarrow \alpha \sim c_{x_0}$ trivial in X