Fundamental group and Covering Spaces

- "functoriality of fundamental group"

X, Y top. spaces $\quad f: X \to Y$ contin. function

$x_0 \in X \quad f(x_0) = y_0 \in Y \quad \exists$ group homomorphism

$$f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

$f_*(\gamma)(t) = f \circ \gamma(t)$ loop based at $f(x_0)$

if $\gamma: [0,1] \to X$ loop
$\gamma(0) = \gamma(1) = x_0$

if $\gamma_1 \sim \gamma_2$ homotopic $h: I \times I \to X$ then
$f_*(\gamma_1) \sim f_*(\gamma_2)$ via $f \circ h$

composition compatible
and $f_1 \circ f_2 \rightsquigarrow (f_1)_* \circ (f_2)_*$

- if X path connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1) \quad \forall x_0, x_1 \in X$
  (write $\pi_1(X)$)

- Covering space $\quad \begin{array}{c} \tilde{X} \\ \downarrow \pi \\ X \end{array}$ projection map (surj., contin.) s.t.
  s.t. $\forall x \in X$ $\exists\, U \subseteq X$ open neighb. of x s.t. $\pi^{-1}(U)$
  is a disjoint union of open sets U_α

$$\pi^{-1}(U) = \bigsqcup_\alpha U_\alpha \quad \text{with} \quad \pi|_{U_\alpha}: U_\alpha \xrightarrow{\cong} U \text{ homeom.}$$

locally a ~~sum~~ disj. union of copies of base
but not globally

Example: $\pi: \mathbb{R} \to S^1 \qquad \pi(x) = \exp(ix) \qquad$ around each $\theta = e^{ix} \in S^1$
$\exists$ interval $(e^{i(x-\delta)}, e^{i(x+\delta)}) = I_\delta$
if δ small union disjoint: $\pi^{-1}(I_\delta) = \bigsqcup_n (x-\delta+2\pi n, x+\delta+2\pi n)$

(2) • covering space $\begin{array}{c}\tilde{X}\\ \downarrow \pi\\ X\end{array}$ and contin. function $f : Y \to X$

• <u>lift of f</u> $= \tilde{f} : Y \to \tilde{X}$ s.t.

$$f = \pi \circ \tilde{f}$$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

• homotopy lifting property: $\begin{array}{c}\tilde{X}\\ \downarrow \pi\\ X\end{array}$ covering space

$F : Y \times I \to X$ homotopy (between $F(y,0) = f_0(y)$ and $F(y,1) = f_1(y)$)
if f_0 admits a lift $\tilde{f}_0$ then $\exists !$ lift $\tilde{F}$
homotopy of $\tilde{f}_0$ ~~to~~ with $\tilde{f}_t$ lift of f_t for all $t \in I$

<u>Pf</u>: $y_0 \in Y$ $\quad U = U(y_0)$ open neighb.

Know $\tilde{F} : U \times \{0\} \to \tilde{X}$ exists given by $\tilde{f}_0$

Suppose have extended $\tilde{F}$ to $U \times [0, \tau]$ for some $\tau > 0$

take $F(U \times [\tau, \tau'])$ for some $\tau' > \tau$ w/ $\tau' - \tau$ small so that
$\exists$ open set $V \subset X$ on which π^{-1} is local homeom.

w/ $F(U \times [\tau, \tau']) \subset V$ $\quad$ ($\pi^{-1}(V) = \coprod_\alpha V_\alpha$ $\quad V_\alpha \underset{\pi}{\cong} V$)

$F(y_0, \tau) \in V$ $\quad$ $\tilde{F}(y_0, \tau) \in V_\alpha$ for <u>one</u> α

take $\pi_\alpha = \pi|_{V_\alpha} : V_\alpha \xrightarrow{\cong} V$

then take $\tilde{F}(y,t) = \pi^{-1}(F(y,t)) \cap V_\alpha$ $\quad$ choose this particular copy in the preimage
$(y,t) \in U \times [\tau, \tau']$

⇝ continue

- When $Y = \{pt\}$ <u>path lifting property</u>

$f : I \to X$ once chosen $\tilde{x}_0 \in \tilde{X}$ $\pi(\tilde{x}_0) = x_0$

$f(0) = x_0$ $\exists!\ \tilde{f} : I \to \tilde{X}$ $\pi \circ \tilde{f} = f$

$\tilde{f}(0) = \tilde{x}_0$

- if $f : I \to X$ and $\tilde{x}_0 \in \tilde{X}$ $\pi(\tilde{x}_0) = x_0$

<u>loop</u> $f(0) = f(1) = x_0$

$\Rightarrow \tilde{f} : I \to \tilde{X}$ <u>path</u> $\tilde{f}(1) \in \pi^{-1}(x_0)$

but in general not same as $\tilde{f}(0) = \tilde{x}_0$

- $$\begin{array}{ccc} \tilde{X} & \ni \tilde{x}_0 & \text{w/ } \pi(\tilde{x}_0) = x_0 \\ \downarrow \pi & & \\ X & \ni x_0 & \end{array}$$

induced morphism

$\pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$

is injective

image $\pi_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$

$\|$

$\{\gamma$ loops$/$htpy in X based at x_0 such that lift $\tilde{\gamma}$ starting at $\tilde{x}_0$ <u>is a loop</u> $\}$

- to see that $\ker(\pi_*) = 1$: an element $[\gamma] \in \ker(\pi_*)$

γ loop w/ $\pi \circ \gamma \underset{\text{htpy}}{\sim} c_{x_0}$ constant loop at x_0

but htpy lifting property $\Rightarrow$ htpy between γ and $c_{\tilde{x}_0}$

- <u>existence of lifts</u> :

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & & \downarrow \pi \\ (Y, y_0) & \xrightarrow[f]{} & (X, x_0) \end{array}$$

condition for existence of lift

$f_*(\pi_1(Y, y_0)) \subseteq \underset{\text{subgroup}}{\pi_*(\pi_1(\tilde{X}, \tilde{x}_0))}$

if lift $\tilde{f}$ have $\pi \circ \tilde{f} = f$ so $\pi_* \circ \tilde{f}_* = f_*$ so OK

④ if $f_*(\pi_1(Y,y_0)) \subseteq \pi_*(\pi_1(\tilde{X},\tilde{x}_0))$ $\quad (*)$

$y \in Y$ γ path y_0 to y ⇝ path lifting:

$f\circ\gamma$ path x_0 to $f(y)$ $\quad \exists!\ \widetilde{f\circ\gamma} : I \to \tilde{X}$

start at $\tilde{x}_0$ w/

$\pi(\widetilde{f\circ\gamma}(1)) = f(y)$

define $\tilde{f}(y) := \widetilde{f\circ\gamma}(1)$

need to know well defined indep of γ used

if $\widetilde{f\circ\gamma}(1)$ and $\widetilde{f\circ\gamma'}(1)$ $\quad \gamma^{-1}\circ\gamma'$ loop in Y at y_0

$f_*([\gamma^{-1}\cdot\gamma'])$ loop in X at x_0 $\quad$ by $(*)$

$\exists\ \alpha$ in $\tilde{X}$ at $\tilde{x}_0$ $\quad \pi_*([\alpha]) = f_*([\gamma^{-1}\cdot\gamma'])$

i.e. $\pi\circ\alpha \underset{\text{htpic}}{\simeq} f\circ(\gamma^{-1}\cdot\gamma')$

through h_t

lifting homotopies $\Rightarrow \exists!\ \tilde{h}_t$ between α and path obtained by

by uniqueness $\tilde{h}_1 = (\widetilde{f\circ\gamma})\cdot(\widetilde{f\circ\gamma'})^{-1}$

htpy of loops at $\tilde{x}_0$ ⇝ $\underbrace{\widetilde{(f\circ\gamma)}(1) = \widetilde{(f\circ\gamma')}(1)}_{\tilde{f}\text{ well defined}}$ $\quad$ concatenation of these paths implies

• ~~lifting property~~ uniqueness of for paths extends to (similar pf)

$$\begin{array}{ccc} & & \tilde{X} \\ & & \downarrow \pi \\ Y & \xrightarrow[f]{} & X \end{array}$$

Y connected

if $\tilde{f}_1, \tilde{f}_2$ lifts of f

agree at one pt

$\Rightarrow \tilde{f}_1 = \tilde{f}_2$ everywhere

- $$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & \tilde{X}_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$ equivalence of covering spaces: homeomorphism ϕ compatible w/ projections $\pi_2 \circ \phi = \pi_1$

- X semi-locally simply connected

  simply connected: path connected with $\pi_1(X) = 1$

  $\forall x \in X$ $\exists\, \mathcal{U}(x)$ open neighb. s.t. $\pi_1(\mathcal{U}(x), x) \xrightarrow{i_*} \pi_1(X, x)$, $i: \mathcal{U}(x) \hookrightarrow X$ inclusion, is trivial

  (Note $\pi_1(\mathcal{U}(x), x)$ need not be trivial but for any (nontrivial) loop in $\underline{\mathcal{U}(x)}$ $\exists$ htpy in $\underline{X}$ to constant loop)

- if X is path connected, locally path connected, semi locally simply connected

  $\Rightarrow \exists$ a covering $\begin{array}{c}\tilde{X}\\ \downarrow \pi\\ X\end{array}$ w/ $\tilde{X}$ simply connected

– Points of $\tilde{X}$ are htpy classes of paths in X starting at x_0 (htpy relative to bdry)

– $\pi: \tilde{X} \to X$ $\quad \pi([\alpha]) = \alpha(1)$ ← well defined

  surjective because X path connected

– topology on $\tilde{X}$:

$$\mathcal{U} = \{ U \subseteq X \text{ open} \mid \pi_1(U) \xrightarrow{i_*} \pi_1(X) \text{ trivial} \}$$

the local path conn + semi local simply conn $\Rightarrow$ $\mathcal{U}$ basis for topology of X

$U \in \mathcal{U}$ $\quad \alpha$ path in X from x_0 to $\overset{\alpha(1)}{\overset{\shortparallel}{x}} \in \mathcal{U}$

$$U_{[\alpha]} = \{ [\alpha \cdot \beta] \mid \beta \text{ path in } \mathcal{U}: \beta(0) = \alpha(1) \}$$

class of htpy rel. to ∂I

(6) $U_{[\alpha]} \subseteq \tilde{X}$ by construction

and $\tilde{\mathcal{U}} = \{ U_{[\alpha]} \mid U \in \mathcal{U}\ \alpha \text{ path} \}$

is a basis for a topology on $\tilde{X}$ (skip explicit checking)

- $\pi : \tilde{X} \to X$ is covering map w/ resp-to these topologies

$\pi : U_{[\alpha]} \to U$ surjective (because U path conn.)

$x \in U \rightsquigarrow$ pick β path in U $\beta(0) = \alpha(1)$ $\beta(1) = x$

so that $\alpha\beta \overset{\pi}{\longmapsto} x$; if β, β' paths same endpts

using $\pi_1(U, \alpha(1)) \xrightarrow{i_*} \pi_1(X, \alpha(1))$ trivial

$\Rightarrow \beta \sim \beta'$

$\Rightarrow [\alpha.\beta] = [\alpha.\beta'] \Rightarrow \pi|_{U_{[\alpha]}} : U_{[\alpha]} \to U$ also injective

$\pi^{-1}(U) = \bigcup U_{[\alpha]}$

coincide $U_{[\alpha]} = U_{[\alpha']}$
(or disjoint $U_{[\alpha]}$)

- $\tilde{X}$ simply connected: it is path connected

α path in X ~~starting~~ at x_0

$\alpha_t(s) = \begin{cases} \alpha(s) & 0 \le s \le t \\ \alpha(t) & t \le s \le 1 \end{cases}$ $\leftarrow$ like α on $[0,t]$ $\leftarrow$ constant $\alpha(t)$ on $[t,1]$

$t \mapsto [\alpha_t]$ map is path from constant $[c_{x_0}]$ path to $[\alpha]$

hence $\tilde{X}$ path conn.

base pt. $\tilde{x}_0 \in \tilde{X}$
$= [c_{x_0}]$

Know $\pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective

so check if $\pi_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is trivial

image of loop: $[\alpha]$ that lifts to $[\alpha_t]$ loop $\alpha_1 = \alpha$ must be $\alpha_1 \sim c_{x_0}$ $\Rightarrow \alpha \sim c_{x_0}$ trivial in