Homotopy classes of paths

path: $\gamma : [0,1] \longrightarrow X$ contin $\gamma(0) = x_0$ $\gamma(1) = x_1$

htpy $\gamma, \gamma' : [0,1] \longrightarrow X$ $H : [0,1] \times [0,1] \longrightarrow X$ $\gamma(0) = \gamma'(0) = x_0$
$\gamma(1) = \gamma'(1) = x_1$

$H(s,0) = \gamma(s)$ $H(s,1) = \gamma'(s)$

if also $H(0,t) = x_0 \ \forall t$ $H(1,t) = x_1 \ \forall t$

htpy through paths from x_0 to x_1

Equivalence relation <u>refl.</u> ok constant $H(s,t) = \gamma(s) \ \forall t$

<u>symm</u> : $H(s, 1-t)$

<u>transitivity</u>: $\gamma_1 \overset{H'}{\sim} \gamma_2$ $\gamma_2 \overset{H''}{\sim} \gamma_3$

at $t = 1/2$

$$H(s,t) = \begin{cases} H'(s, 2t) & 0 \le t \le \frac{1}{2} \\ H''(s, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$H'(s,1) = \gamma'(s) = H''(s,0)$

$[\gamma]$ equivalence classes of paths in X up to homotopy

Product structure:

γ path from x_0 to x_1 γ' path from x_1 to x_2

$$\gamma * \gamma'(s) = \begin{cases} \gamma(2s) & 0 \le s \le \frac{1}{2} \\ \gamma'(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

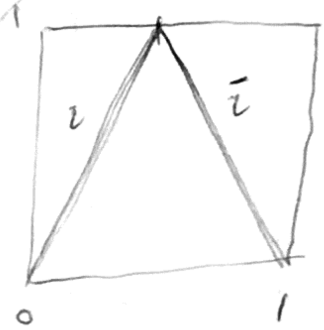
this product is compatible with equivalence by htpy

$[\gamma]*[\gamma'] = [\gamma*\gamma']$ $\qquad$ $\gamma \overset{H}{\sim} \tilde{\gamma}$ $\quad$ $\gamma' \overset{H'}{\sim} \tilde{\gamma}'$

using homotopy

$$\tilde{H}(s,t) = \begin{cases} H(2s,t) & 0 \le s \le \frac{1}{2} \\ H'(2s-1,t) & \frac{1}{2} \le s \le 1 \end{cases}$$

$$\gamma * \gamma' \overset{\tilde{H}}{\sim} \tilde{\gamma} * \tilde{\gamma}'$$



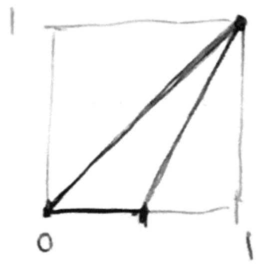
$\gamma * \bar{\gamma} = \begin{cases} \gamma \circ \iota \\ \gamma \circ \bar{\iota} \end{cases}$

Properties of this product:

- identity elements (left/right)

γ path from x_0 to x_1 $\qquad$ e_{x_i} = constant path at x_i

$$[\gamma]*[e_{x_1}] = [\gamma] \quad \& \quad [e_{x_0}]*[\gamma] = [\gamma]$$



$$I \xrightarrow{\gamma} X$$

precomp. w/ htpy between paths in I

- inverses $\forall [\gamma] \; \exists [\bar{\gamma}]$ reverse orient. path

s.t. $[\gamma]*[\bar{\gamma}] = [e_{x_0}]$ and $[\bar{\gamma}]*[\gamma] = [e_{x_1}]$

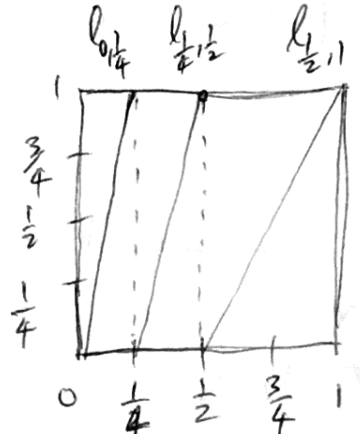
htpy of paths in I between $\bar{\iota} \circ \iota$ and constant path at 0 composed w/ γ gives htpy to e_{x_0} $\quad$ similar for second case

- associativity

$$([\gamma_1]*[\gamma_2])*[\gamma_3] = [\gamma_1]*([\gamma_2]*[\gamma_3])$$

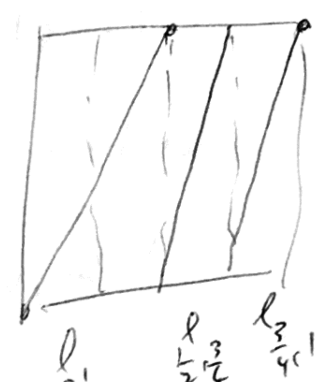
htpy class of path

$$\left\{\begin{array}{l} \gamma_1 \circ l_{0,1/4} \\ \gamma_2 \circ l_{1/4,1/2} \\ \gamma_3 \circ l_{1/2,1} \end{array}\right.$$

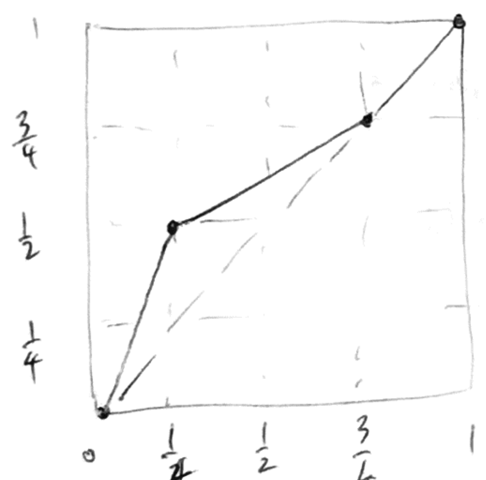


$= \underline{\gamma} \circ \underline{l}$

$$\left\{\begin{array}{l} \gamma_1 \circ l_{0,\frac12} \\ \gamma_2 \circ l_{\frac12,\frac34} \\ \gamma_3 \circ l_{\frac34,1} \end{array}\right.$$



$= \underline{\gamma} \circ \underline{\tilde{l}}$



$p: I \to I$ $\qquad p \circ l = \tilde{l}$

H = homotopy of paths in I between p and $p_0(s) = s$

$\underline{\gamma} \circ H \circ \underline{l}$ is htpy between $\underline{\gamma} \circ \underline{l}$ and $\underline{\gamma} \circ \tilde{l}$

so between $(\gamma_1 * \gamma_2) * \gamma_3$ and $\gamma_1 * (\gamma_2 * \gamma_3)$

- Set of htpy classes of paths in X with $*$ operation

<u>Groupoid</u> = a category where every morphism is an isomorphism

- $\Pi_1(X)$

$\Pi_1(X)$ groupoid

$\mathrm{Obj}(\Pi_1(X)) = \{ x \in X \}$ pts of X

$\mathrm{Mor}_{\Pi_1(X)}(x,y) = \{ [\gamma] \text{ htpy classes} \mid \gamma : [0,1] \to X, \ \gamma(0) = x \ \ \gamma(1) = y \}$

- composition of morphisms

$$\mathrm{Mor}_{\Pi_1(X)}(x,y) \times \mathrm{Mor}_{\Pi_1(X)}(y,z) \longrightarrow \mathrm{Mor}_{\Pi_1(X)}(x,z)$$
$$[\gamma_1] \qquad [\gamma_2] \longmapsto [\gamma_1] * [\gamma_2]$$

associativity of composition ok

- identity morphism = constant paths $[e_x]$

- groupoid: all morphisms are invertible w/ inverse $[\bar{\gamma}]$

$\Pi_1(X)$ = fundamental groupoid of X

Π_1 functor $\qquad \Pi_1 : \mathrm{Top} \longrightarrow \mathrm{Grpoids}$

$$X \longmapsto \Pi_1(X)$$

$$f : X \to Y \mapsto f_* : \Pi_1(X) \to \Pi_1(Y)$$

<u>Grpoids</u> category of groupoids
morphisms = functors between groupoids

$f: X \to Y$ cont. function

$f_*: \Pi_1(X) \longrightarrow \Pi_1(Y)$

$\gamma: [0,1] \to X$ $\qquad$ $f \circ \gamma: [0,1] \longrightarrow Y$

if $\gamma \overset{H}{\sim} \gamma'$ then $f \circ \gamma \overset{f \circ H}{\sim} f \circ \gamma'$ $\qquad$ so $f_*: [\gamma] \mapsto [f \circ \gamma]$

$$f_*\big(\underbrace{[\gamma_1] * [\gamma_2]}\big) = f_*[\gamma_1] * f_*[\gamma_2]$$

$$f \circ \begin{cases} \gamma_1 \circ l_{0,\frac{1}{2}} \\ \gamma_2 \circ l_{\frac{1}{2},1} \end{cases} = \begin{cases} (f \circ \gamma_1) \circ l_{0,\frac{1}{2}} \\ (f \circ \gamma_2) \circ l_{\frac{1}{2},1} \end{cases}$$

So f_* is a functor from $\Pi_1(X)$ to $\Pi_1(Y)$

objects: $x \in X \mapsto f(x) \in Y$

morphisms $[\gamma]$ $\gamma(0)=x$, $\gamma(1)=y$ $\longmapsto$ $[f \circ \gamma]$ from $f(x)$ to $f(y)$

compatible w/ composition of morphisms & id morphisms

A group G is a groupoid w/ a single object

Fundamental group

$x_0 \in X$ a chosen pt.

$$\pi_1(X, x_0) = \{ [\gamma] : \gamma : [0,1] \to X \quad \gamma(0) = \gamma(1) = x_0 \}$$

homotopy classes of loops in X

(not all paths: same initial & final pt.)

w/ same prod $*$ is a group

Not a functor from Top to Groups because of dep. on x_0

Top_* = category of pointed topological spaces

objects (X, x_0) pairs of top space $(X, \mathcal{T}_X)$ and a choice of a base pt. $x_0 \in X$

morphisms

$$Mor_{Top_*}((X, x_0), (Y, y_0)) = \{ f : X \to Y \text{ contin. s.t. } f(x_0) = y_0 \}$$

Different categorical properties w/ resp. to Top

e.g. in Top_* $(\{*\}, *)$ is a "zero object" both initial & terminal

but Top does not have 0-obj.

π_1 functor $\pi_1 : Top_* \longrightarrow Groups$

Fundamental group

More on pointed topological spaces Top_*

- <u>coproduct</u> (wedge product)

$$(X, x_0) \sqcup (Y, y_0) = X \sqcup Y / \sim$$

disj. union = coprod in Sets (and in Top)

equiv. rel. $x_0 \sim y_0$ basept identified

notation: $(X, x_0) \vee (Y, y_0)$

e.g. $(S^1, 1) \sqcup (S^1, 1)$

one-pt-union of two circles

- <u>product</u> $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$

(just ordinary prod of top. sp. w/ ~~basept~~ basept.)

univ. property of coprod

$\exists$ <u>inclusions</u> $(X, x_0) \overset{i_X}{\hookrightarrow} (X, x_0) \vee (Y, y_0) \overset{i_Y}{\hookleftarrow} (Y, y_0)$

(as morphisms of pointed top spaces) s.t.

given maps in Top_*
$f : (X, x_0) \to (Z, z_0)$ $(f(x_0) = z_0)$
$g : (Y, y_0) \to (Z, z_0)$ $(g(y_0) = z_0)$

then they can be glued together to map

$f \vee g : (X, x_0) \vee (Y, y_0) \to (Z, z_0)$ since they have $f(x_0) = g(y_0)$

• <u>smash product</u> in Top_*

$$(X,x_0)\wedge(Y,y_0) = X\times Y \big/ X\times\{y_0\}\cup\{x_0\}\times Y$$

$$= X\times Y \big/ X\vee Y$$

better categorical property that just prod
(<u>product/Hom adjunction</u>)

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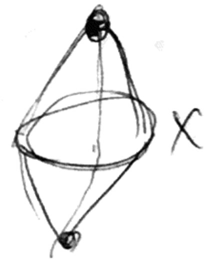
reduced suspension $\Sigma X = X\wedge S^1 = X\times S^1 \big/ X\times\{1\}\cup\{x_0\}\times S^1$

suspension $SX = X\times I \big/ \sim$ $\quad (x,0)\sim(x',0)$, $(x,1)\sim(x',1)$



respectively in Top_* and in Top

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based loop $\Omega X = \underbrace{Mor_{Top_*}(S^1,(X,x_0))}$

this is a set: need a <u>topology</u> to think of ΩX as object in Top_*
(based pt = constant loop at x_0)

"compact-open topology" on $Mor_{Top_*}(S^1,(X,x_0))$

subbasis of topology

sets $S(K,U)$ $\quad K\subset S^1$ compact, $U\subset X$ open

$$S(K,U) = \{f\in Mor_{Top_*}(S^1,X) \mid f(K)\subset U\}$$

Suspension – loop adjunction

$$\mathrm{Top}_* \underset{\Sigma}{\overset{\Omega}{\rightleftarrows}} \mathrm{Top}_*$$

$$\mathrm{Mor}_{\mathrm{Top}_*}(\Sigma X, Y) \cong \mathrm{Mor}_{\mathrm{Top}_*}(X, \Omega Y)$$

$f : \Sigma X \to Y$ take $x \in X$ and $\{x\} \times I$ set in $X \times I$

$\{x\} \times S^1$ in pted case

image under f of this $\{x\} \times S^1$ is a loop in Y based at $f(x)$

$\gamma : S^1 \to Y$ $\quad \gamma(t) = f(x, t)$

$\cup$

$\{x\} \times S^1$

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homotopy classes of maps (compatible w/ homotopies)

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

in particular higher homotopy groups

$\pi_n X := [S^n, X]$ $\quad$ n-sphere $S^n \simeq \Sigma S^{n-1}$

$$= [S^{n-1}, \Omega X] = \pi_{n-1} \Omega X$$