fact: 2-cell σ = homeom. to closed 2-disk $\overline{D}^2 \subseteq \mathbb{R}^2$

$$K = \bigcap_{i=1}^{\infty} \sigma_\nu \qquad \sigma_\nu \subset \mathring{\sigma}_{\nu-1} \qquad \nu \in \mathbb{N}$$

"cellular set" in a top. surface X

then $M \underset{\text{homeom.}}{\cong} M/K$

M top. surface $C \subseteq M$ connected and union of simple closed curves

$$C = \bigcup_{i=1}^{n} C_i$$

if A compact totally disconnected subset of C then A contained in interior of a closed 2-cell σ in X

Pf: A compact & tot disconn
$\Rightarrow \exists$ a subarc γ of C_1 that does not intersect A

$C_1 \setminus \gamma$ can be <u>thickened</u> to contain a closed 2-cell σ_1 s.t. $\mathring{\sigma}_1 \supset A \cap C_1$

thickening of an arc 

γ arc in X 2-mfld $\Rightarrow$ contained in interior of a 2-cell (disjoint from any preassigned compact set in γ^c)

next step: if 2-cell σ_k s.t.

$$A \cap \bigcup_{i=1}^{k} C_k \subseteq \mathring{\sigma}_k \quad \text{obtained}$$

then $C_{k+1} \setminus \sigma_k$ is union of a collection of at most countably many disjoint open arcs γ_ℓ w/ endpts in $\partial\sigma_k$

each γ_ℓ has some subarc γ_ℓ' not intersecting A

$A \cap \gamma_\ell' = \emptyset$

then $\gamma_\ell \setminus \gamma_\ell'$ two arcs $\rightsquigarrow$ thicken

2 closed cells $\sigma_{\ell,1}$ $\sigma_{\ell,2}$

w/ $\sigma_{\ell,1} \cap \sigma_{\ell,2} = \emptyset$

$\sigma_{\ell,i} \cap \gamma_{\ell'} = \emptyset \quad \ell \neq \ell'$

$\sigma_{\ell,i} \cap \sigma_k$ = closed 2-cells

then these ensure that

$\sigma_{k+1} = \sigma_k \cup \bigcup_\ell \sigma_{\ell,1} \cup \sigma_{\ell,2}$ still has the topology of a 2-cell

and $A \cap \bigcup_{i=1}^{k+1} C_k \subseteq \mathring{\sigma}_{k+1}$

Then use this to show that
<u>every 2-surface can be triangulated</u>

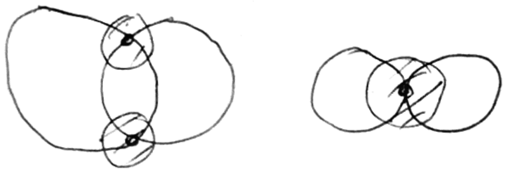
- Start by covering X w/ a finite collection of open disks (open sets homeom. to open disks) w/ closures homeom. to closed disks $D_1, \ldots, D_n$

curves $C_i = \partial D_i$

in union $\bigcup_{i=1}^{n} C_i$ look at set of "singular pts" A

i.e. those that do not have a neighb. homeom to interval in C

Compact & totally discon.



can assume covering $\{\mathring{D}_1, \dots, \mathring{D}_n\}$ minimal
i.e no smaller subcollection covers all of X

$\Longrightarrow$ C is connected

avoids this kind of situation

- previous observation $\Longrightarrow$ $\exists$ a closed 2-cell σ in X s.t. $A \subseteq \mathring{\sigma}$

- $X \setminus C$ union of an at most countable collection of disjoint open 2-cells
(w/ closures that are 2-cells in X)

follows from Jordan-Schoenflies theorem

- $C \setminus \sigma$ at most countable union of mutually disjoint open arcs w/ endpts on $\partial\sigma$.

- $\sigma \subseteq U \subseteq \cancel{X}$ $\Longrightarrow$ σ is cellular
open 2-cell (by thickening $\partial\sigma$) (Jordan-Schoenflies)
$\Downarrow$
$X \cong X/\sigma$

- take $\mathcal{R} \subset X/\sigma$ image in quotient of $\overline{C \setminus \sigma}$

  ~~...~~ $\mathcal{R}$ = one-pt. union of at most countable coll. of simple closed curves

  w/ only one singular pts where σ collapsed

  y

- $X \setminus \mathcal{R} \underset{\text{homeom}}{\cong} X \setminus (C \cup \sigma)$

- a 2-cell neighb. V of y

  Contains all but fin. many of simple closed curves in $\mathcal{R}$

  by compactness of $\overline{C - \sigma}$ $\Rightarrow$ compactness of $\mathcal{R}$

  (otherwise would obtain an ∞ covering of $\mathcal{R}$ from which no fin. covering can be extracted)

- obtain a cellular set in V one-pt-union of closed disks bound by curves in $\mathcal{R}$ contained in V joined at pt. y

  T

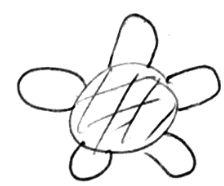
- pass to $X/T \cong X$ left w/ fin. many

  $\tilde{\mathcal{R}}$ =

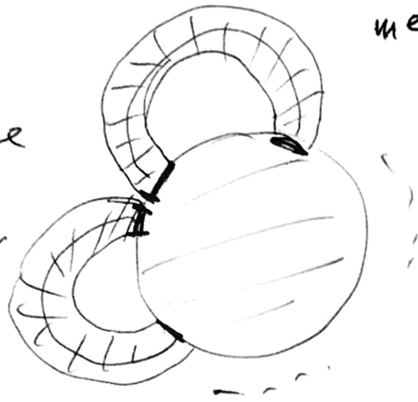
  w/ complement = fin. many open 2-cells

- pick small closed 2-cell e each curve meets ∂e in 2 pts

- then $e \cup \tilde{R}$ = a disk w/ fin. many mutually disjoint closed arcs meeting ∂e in two pts



- enclose each arc in a 2-disk



N

← this is a triangulable 2-manifold w/ boundary

- each component of $M \setminus N$: extend the triangulation because each component is homeo to a disk of the original fin. cover.