Every 2-dim topological surface can be triangulated

Preliminary result needed:

Jordan-Schoenflies theorem

1) a simple closed curve $J$ in $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two regions

2) $\exists$ homeom. $f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f(J) =$ circle

Note: $J$ separates $\mathbb{R}^2$ means $\mathbb{R}^2 \setminus J = U \cup V$ two connected components $U, V \neq \emptyset$, $U \cap V = \emptyset$

simple closed curve: means $h: S^1 \to \mathbb{R}^2$ $\quad h: S^1 \xrightarrow{\simeq} h(S^1) = J$ homeom.
a homeomorphic image of $S^1$ in $\mathbb{R}^2$

• first: $\mathbb{R}^2 \setminus J$ has at least two connected components:

this uses Borsuk separation

A compact subset of Euclidean $\mathbb{R}^{n+1}$ has $\mathbb{R}^{n+1} \setminus A$ not connected (separated) iff $\exists$ $f: A \to S^n$ that not null-homotopic

homotopic functions

$f_0, f_1 : X \to Y$ continuous

are homotopic if $\exists\ \Phi : X \times I \to Y$ contin.

w/ $I = [0,1]$ Euclidean s.t. $\Phi(x,0) = f_0(x)$

$\Phi(x,1) = f_1(x)$

(can be deformed continuously one into the other)

null-homotopic = homotopic to a constant map

$\boxed{\Rightarrow :}$ $A^c$ separated $\Rightarrow \exists$ non-null homotopic $f : A \to S^n$

First observation: $A \subset \mathbb{R}^{n+1}$ compact

then $\mathbb{R}^{n+1} \setminus A$ has exactly one unbounded component

because A compact so bounded (in metric space $\mathbb{R}^{n+1}, d = d_{Eucl.}$)

$\Rightarrow \exists\ B_d(0,r)$ s.t. $A \subseteq \overline{B_d(0,r)}$ inside some closed ball

$\Rightarrow \overline{B_d(0,r)}^c \subseteq A^c$ connected hence inside one of the connected components of $A^c$ (the unbounded component)

$\Rightarrow$ So if $A^c$ there is at least one bounded connected component $U$

take $x \in U$ consider Borsuk map

$$\beta_x : \mathbb{R}^{n+1} \setminus \{x\} \to S^n$$

$$y \mapsto \frac{y-x}{\|y-x\|}$$ cont. map

restrict $\beta_x$ to A

Second observation:

the Borsuk map $\beta_x\big|_A : A \to S^n$

is a continuous map that is not null-homotopic

to show this

first show that

if $A \subseteq \mathbb{R}^{n+1}$ closed
a contin. function $f: A \to S^n$ is null-homotopic
iff it can be extended continuously to $f: \mathbb{R}^{n+1} \to S^n$

(in fact same true for $A \subseteq \mathbb{R}^k$ any $k$ and $f: A \to S^n$)

Pf of this: if $f \simeq 0$ ($f$ homotopic to constant function)

if $f_0 \overset{\varphi: A\times I \to S^n \text{ htpy}}{\simeq} f_1 : A \to S^n$ and $f_0$ extendable to $\mathbb{R}^{n+1}$ then so also $f_1$

$F_0 : \mathbb{R}^{n+1} \to S^n \quad F_0|_A = f_0$

take $\Phi_0 : \mathbb{R}^{n+1} \times \{0\} \cup A \times I \to S^n$

with $\Phi_0(x,0) = F_0(x) \qquad \Phi_0(a,t) = \varphi(a,t)$

want to extend this $\Phi_0$ to $\mathbb{R}^{n+1} \times I$

Construct as $\Phi(x,t) = \widetilde{\Phi}_0(x, t\, h(x))$

Urysohn lemma on each coord.

where $\widetilde{\Phi}_0$ is an extension of $\Phi_0$ to some open neighb $\mathcal{U}$ of $\underbrace{\mathbb{R}^{n+1} \times \{0\} \cup A \times I}_{\text{closed set in } \mathbb{R}^{n+1}\times I}$

obtained by first extending to some $\hat{\Phi}_0 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ then $\widetilde{\Phi}_0 = \dfrac{\hat{\Phi}_0}{\|\hat{\Phi}_0\|}$ on $\mathcal{U}$ where $\hat{\Phi}_0 \neq 0$

and $h: \mathbb{R}^{n+1} \to I$ Urysohn function w/

$h|_A \equiv 1 \qquad h|_{\mathbb{R}^{n+1} \setminus V} \equiv 0 \qquad \underset{\text{open}}{V \supset A}$ s.t. $V \times I \subseteq \mathcal{U}$

(4)

if $f$ extendable to $\mathbb{R}^{n+1}$

then $F: \mathbb{R}^{n+1} \longrightarrow S^n$ $\quad F|_A = f$

$\mathbb{R}^{n+1}$ linear space

$v \in \mathbb{R}^{n+1} \Rightarrow \lambda v \in \mathbb{R}^{n+1}$ $\quad \lambda \in \mathbb{R}$

$\phi: \mathbb{R}^{n+1} \times I \longrightarrow S^n$ $\quad \phi(v,t) = F(tv)$

is homotopy $F \simeq 0$ constant map $v \longmapsto F(0)$

$\Rightarrow \phi|_{A \times I}$ then htpy of $f$ to constant map

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then to show Borsuk map $\beta_x|_A : A \to S^n$

is not-null-homotopic

show cannot be extended to $\mathbb{R}^{n+1}$

in fact show cannot be extended to $A \cup \mathcal{U}$

bounded conn. comp. of $A^c$

Suppose have extension

$F: A \cup \mathcal{U} \to S^n$ $\quad F|_A = \beta_x|_A$

take ball around $x$ $\quad B_d(x,R)$ s.t.

$A \cup \mathcal{U} \subseteq B_d(x,R)$ $\quad$ (possible since both $A, \mathcal{U}$ bounded)

and obtain map $\quad g: \overline{B_d(x,R)} \longrightarrow \partial B_d(x,R) \cong S^n_R$

$g(y) = y + R\,\dfrac{y-x}{\|y-x\|}$ $\quad y \in \overline{B_d(x,R)} \setminus \mathcal{U}$ $\quad$ and $\quad = y + R \cdot F(x)$ $\quad y \in \overline{\mathcal{U}}$

continuous (same values on $A$) $\quad$ identity on $\partial B_d(x,R)$

this gives a continuous surjective map

$$g : \overline{B_d(x,R)} \to \partial B_d(x,R)$$

that is identity on boundary

i.e. up to scaling and transl. homeomorphisms

a cont. map $\overline{B}^{n+1} \xrightarrow{g} S^n$ unit sphere in $\mathbb{R}^{n+1}$

closed unit ball in $\mathbb{R}^{n+1}$

that is identity on $S^n = \partial \overline{B}^{n+1}$

<u>Other fact</u>: there is no continuous retraction of the ball to the sphere

<u>Brouwer's theorem</u>

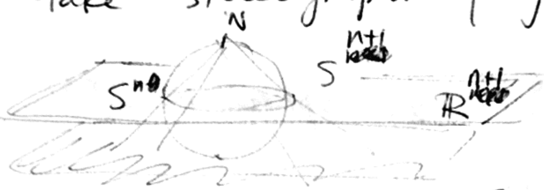
(later)

$\boxed{\Leftarrow}$ $\exists f: A \to S^n$ non-null homotopic $\Rightarrow$ $A^c$ separated

if $A^c$ not separated (i.e. $A^c$ connected)
only one component in $A^c$ = unbounded component

then any $f: A \to S^n$ is extendable to $F: \mathbb{R}^{n+1} \to S^n$

take stereographic projection




$\pi: S^{n+1} \setminus \{N\} \xrightarrow{\sim} \mathbb{R}^{n+1}$ homeom.

$\pi^{-1}(A) \subset S^{n+1}$ compact

so extension property becomes
$f: A \to S^n$ extension to $S^{n+1} \setminus \{N\}$
$A \subset S^{n+1} \setminus \{N\}$

To construct these extensions

use that spheres (in any dimension)
can be triangulated (by simplexes of the appropriate dimension)

first extend from $A$
to union of all $(n+1)$-simplexes in $S^{n+1}$
intersect $A$ (using convex structure of simplexes)

then in each $(n+1)$-simplex ~~not contained~~ contained in $A^c$
map vertices to any pts in $S^n$
edges to arcs between vertices
etc ... map $n$-dim faces
from interior of each $(n+1)$-simplex
remove one pt, project to boundary
then map boundary

$\Rightarrow$ extension $F$ to $S^{n+1} \setminus \bigcup_i p_i$ one removed pt. from each $(n+1)$-simplex in $A^c$

$A^c$ has only one conn. component

$\Rightarrow$ $A^c$ open set in $S^{n+1}$ is path connected
(open set in $\mathbb{R}^{n+1}$ w/ Eucl. distance)
-- cover w/ sequence of balls

take path going through all pts $p_i$ and ending at North pole $N$ of $S^{n+1}$

$\gamma : [0,1] \longrightarrow S^{n+1}$ $\quad \gamma(0) = p_0, \dots, \gamma(1) = N$

$\gamma(I)$ compact cover w/ fin. many balls $B_0, \dots, B_n$ $\quad \bar{B}_i \subset A^c$

$B_i \cap B_j \neq \emptyset$ $\quad p_0 \in B_0$ $\quad N \in B_n$

replace $F|_{B_0 \setminus \{p_i \in B_0\}}$ w/ $F|_{\partial B_0} \circ \pi : B_0 \setminus \{x_1\} \longrightarrow S^n$ $\quad$ ($B_0 \setminus \{x_1\} \xrightarrow{\pi} \partial B_0 \xrightarrow{F} S^n$)

$x_1 \in B_0 \cap B_1$

etc. replace all $p_i \in B_0$ w/ single $x_1 \in B_1$
--- single $x_n = N$ in $B_n$

$\Rightarrow$ obtain extension $F$ to $S^{n+1} \setminus \{N\} \longrightarrow S^n$
$S^{n+1} \setminus \{N\} \cong \mathbb{R}^{n+1}$

$\Rightarrow$ so $f$ must be null homotopic

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conclusion

having proved Borsuk separation
( $A^c$ has at least two components )

then $\Rightarrow$

Want to know <u>exactly</u> two components

Note: $J \subset \mathbb{R}^2$ simple closed curve : homeomorphic image of $S^1 \xhookrightarrow[\text{homeom.}]{h} \mathbb{R}^2$ $\quad J = h(S^1)$

suppose all $f : J \to S^1$ null-homotopic $\Rightarrow$ $f \circ h$ null-htpic : $S^1 \to S^1$
but $S^1$ separates $\mathbb{R}^2$ $\Rightarrow$ non-null-htpic $S^1 \to S^1$

suppose more than two components in $\mathbb{R}^2 \setminus J$

e.g. at least three $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$

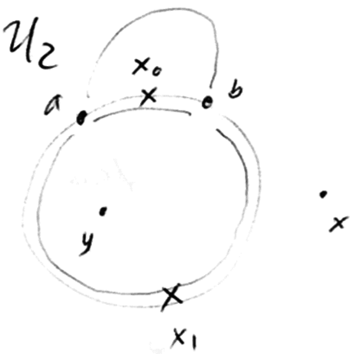
take $a \neq b \in J$ both accessible from paths from $\mathcal{U}_2$

$\gamma$ arc in $\mathcal{U}_2$ joining $a, b$.

$\gamma : [0,1] \to \mathbb{R}^2$

$\gamma(0) = a$ , $\gamma(1) = b$

$\gamma((0,1)) \subset \mathcal{U}_2$




on $J$ two arcs

$\gamma_0, \gamma_1$ w/ endpts. $a, b$

take pts $x_0, x_1$ on each of these

$\alpha = \text{arc}(a x_0 b) \cup \gamma$

$\beta = \text{arc}(b x_1 a) \cup \gamma$

take $x \in \mathcal{U}_0$
$y \in \mathcal{U}_1$

Claim $\beta, \alpha \subset \mathbb{R}^2$ compact & don't separate $x, y$ & $\alpha \cap \beta$ connected $\Rightarrow \alpha \cup \beta$ does not separate $x, y$

1) $\alpha$ does not separate $x, y$ because $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \{x_1\}$ connected set in $\alpha^c$ containing both $x, y$

2) $\beta$ does not separate $x, y$ because $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \{x_0\}$ connected set in $\beta^c$ containing both $x, y$

3) $\alpha \cap \beta = \gamma$ connected

$\Rightarrow \alpha \cup \beta$ does not separate $x, y$ but by hypoth. $J \subset \alpha \cup \beta$ separates $x, y$