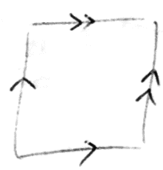
Topological classification of surfaces

Topological surface X : compact connected Hausdorff space w/ countable basis s.t. locally homeom. to $\mathbb{R}^2$ w/ Euclidean top.

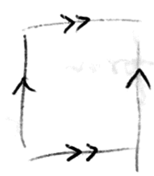
$\forall\ x \in X \quad \exists\, \mathcal{U}(x)$ and $\varphi_x : \mathcal{U}(x) \xrightarrow{\sim} U \subseteq \mathbb{R}^2$ homeom. to an open set in $\mathbb{R}^2$

(2-dimensional topological manifold)

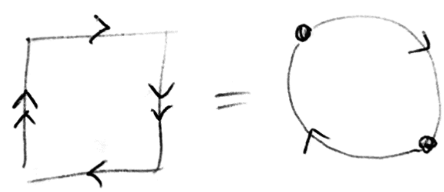
Example: Sphere S^2 obtained as quotient



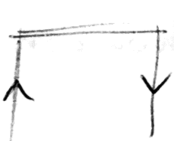
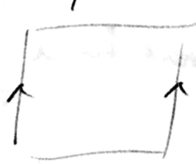
torus $\mathbb{T}^2$ obtained as quotient



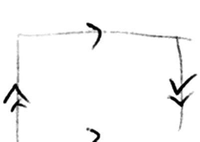
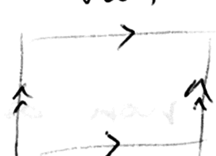
projective plane $\mathbb{P}^2$ obtained as quotient



(Note : Möbius band vs. cylinder)

Klein bottle torus

 vs. 

Connected sum operation: X_1, X_2 top. surfaces

$X_1 \# X_2$

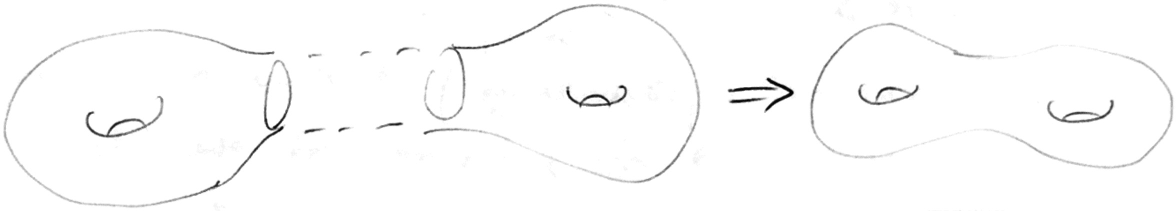
pick $x_1 \in X$ $\quad x_2 \in X_2$ $\quad \exists\, D(x_1)$ $\quad D(x_2)$

open neighb. homeom. to disks in $\mathbb{R}^2$

take $X_1 \setminus D(x_1)$ $\quad X_2 \setminus D(x_2)$

$\partial(X_1 \setminus D(x_1)) \simeq S^1$ $\quad \partial(X_2 \setminus D(x_2)) \simeq S^1$

quotient by identifying these two circles



Note:
$X \# S^2 \cong X$

$S^2 \setminus D \simeq D$

Surfaces in standard form

$X = \#_g T^2 \# \#_h \mathbb{P}^2$

g-connected sums of tori
& h-connected sums of proj. spaces

Classification theorem

X topological surface: then X homeomorphic to one of

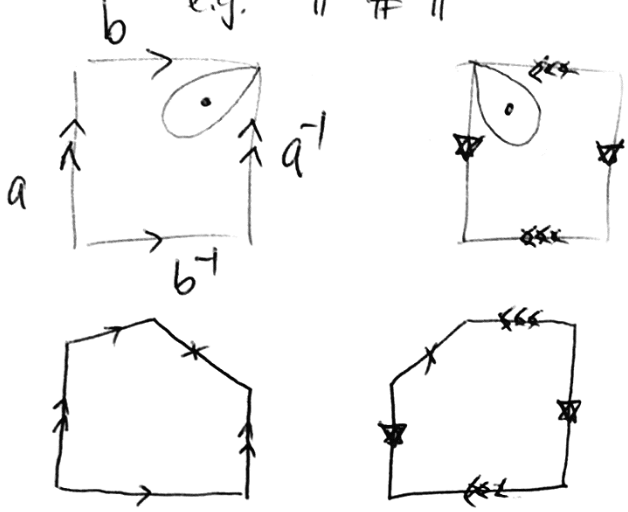
1) $X \cong S^2$
2) $X \cong \#_g T^2$

} orientable

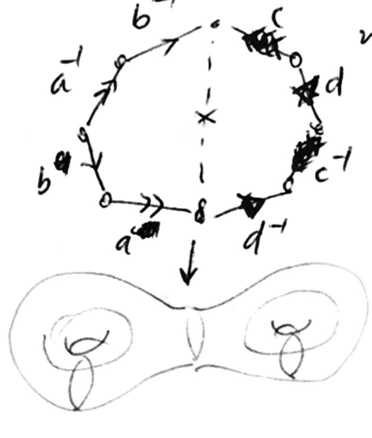
3) $X \cong \#_h \mathbb{P}^2$ non-orientable

1] planar gluing diagrams for connected sums

e.g. $\mathbb{T}^2 \# \mathbb{T}^2$



8-sides polygon w/ identif pairwise of edges



quotient surface

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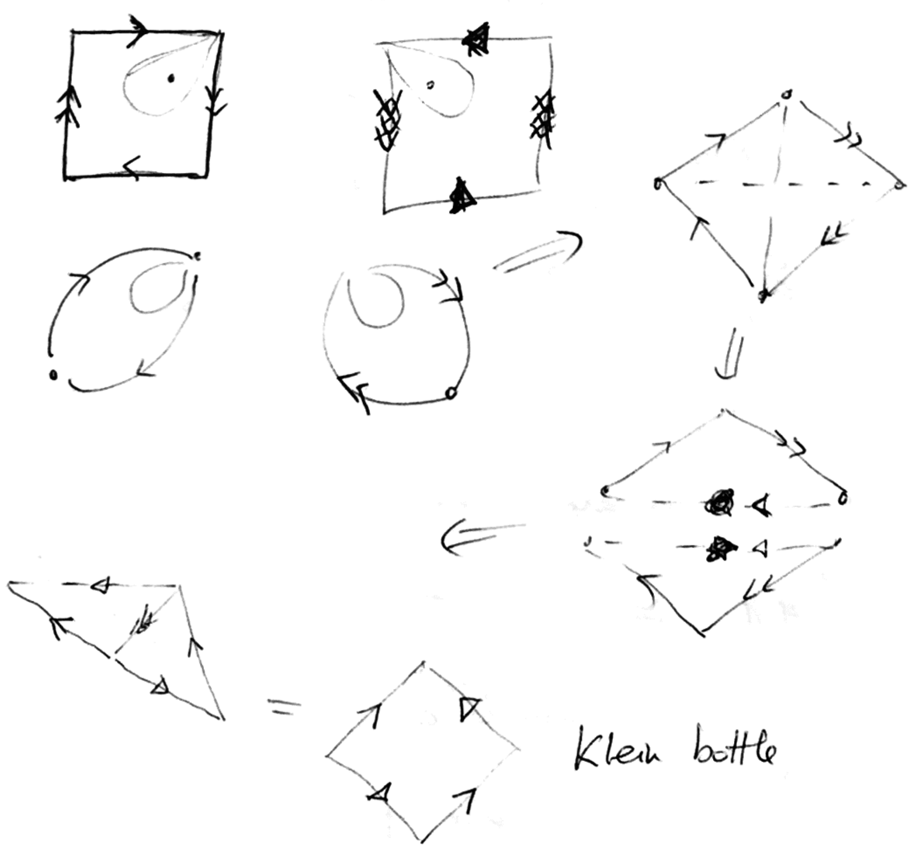
e.g. connected sum $\mathbb{P}^2 \# \mathbb{P}^2$ = Klein bottle

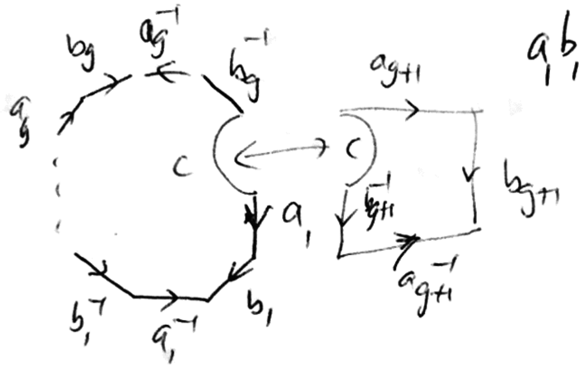


Klein bottle

Standard planar gluing diagrams

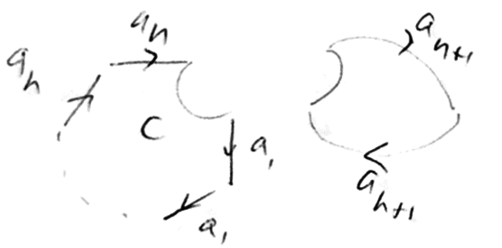
$\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g\text{-times}}$ 4g polygon w/ identification

$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$



$\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{h}$ 2h polygon w/ identification

$a_1 a_1\, a_2 a_2 \cdots a_h a_h$



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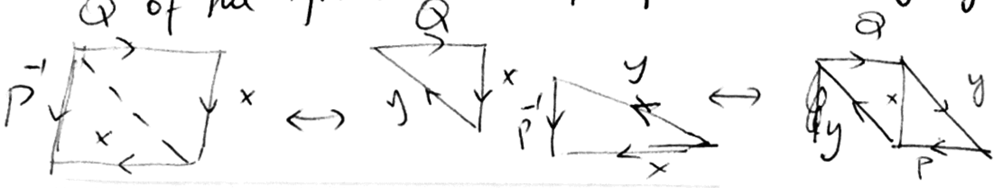
identifications: there are homeomorphisms

$$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \simeq \mathbb{T}^2 \# \mathbb{P}^2$$

aabbcc $\qquad$ $aba^{-1}b^{-1}cc$

general fact: can replace a sequence of gluing instructions of the from $xxP^{-1}Q$ with $yPyQ$



Note: So space $\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{h \text{ odd}}$ homeom. to $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{\frac{h-1}{2}} \# \mathbb{P}^2$

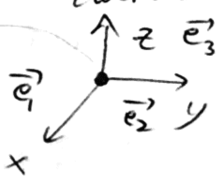
$\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{h \text{ even}}$ homeom. to $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{\frac{h-2}{2}} \# \text{Klein}$

So can give list of "standard" surfaces either in terms of $\#_h \mathbb{P}^2$ or in terms of $\#_g \mathbb{T}^2 (\# \mathbb{P}^2)$

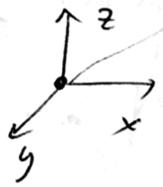
$\#_g \mathbb{T}^2$

Orientation: X is orientable

locally $\mathcal{U}(x) \simeq \mathcal{U} \subseteq \mathbb{R}^2$ local coord (x,y) locally view in $\mathbb{R}^3$ w/ choice of normal direction $\vec{n}$ so (x,y,z) oriented "clockwise"

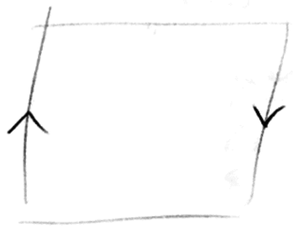


orientable X if cannot move continuously this choice of oriented vectors w/ $x \in X$ and obtain opposite orient!



non orientable if possible

Möbius band



non orientable

cylinder

orientable

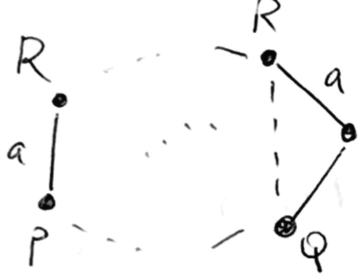
$\boxed{2}$ Second step for classification:

manipulation of planar gluing diagrams that gives homeomorphic surfaces

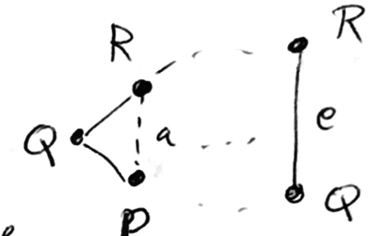
1) Can always reduce a gluing polygon to one where all vertices identify to same point.

~~suppose~~ label vertices by pt. they identify to

suppose



$\longleftrightarrow$



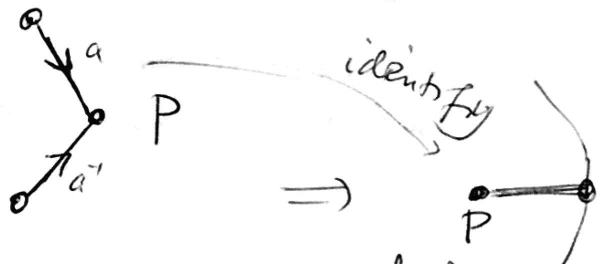
w/ $P \neq Q$

(R may be same or not of one of these)

removed one P-vertex
added one Q-vertex

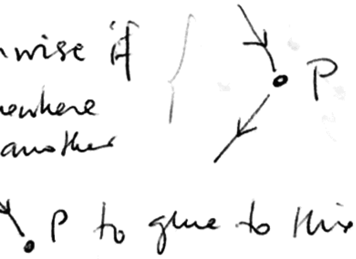
repeat until eliminate all but one P-vertex

diagram then looks like this:



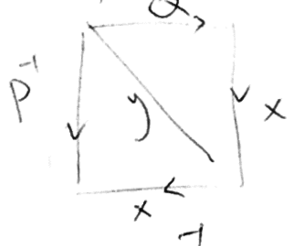
and eliminate P

otherwise if somewhere need another

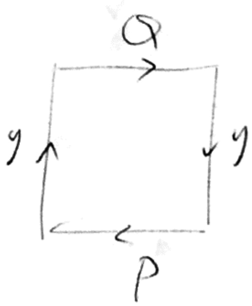


P to glue to this one

2) already observed more



$\simeq$



$xxP^{-1}Q \simeq y\,P\,y\,Q$

3) also similar move:

$$xPQx^{-1}R \simeq yQPy^{-1}R$$

4) note that sequences of gluing instructions can be cyclically permuted so $xxP^{-1}QS = SxxP^{-1}Q \simeq yPyQS$
$= SyPyQ$
etc.

5) Use these moves (move 2) to make all repetitions appear consecutively: xx
(replace a seq. $SxPxQ$ by $SyyP^{-1}Q$)

- So reduce a polygon w/ sides pairwise identified w/ one with some aa pairings (adjacent) and some aa^{-1} pairings: $aba^{-1}b^{-1}$ ---- in any order

two sides a,b crossed pair if occur as

$$PaQbRa^{-1}Sb^{-1}T \quad \text{(up to cyclic perm.)}$$

- can always reduce to a presentation where crossed pairs appear as $aba^{-1}b^{-1}$
  - (3) implies in partic. $xPQx^{-1} \simeq x_1QPx_1^{-1}$
  - $P\underbrace{xQ}\underbrace{yRx^{-1}}Sy^{-1}T \longrightarrow Px_1\underbrace{yRQx_1^{-1}}Sy^{-1}T$

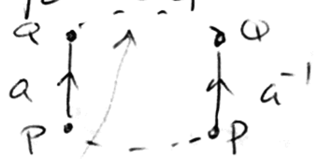
$\longrightarrow \quad P\, x_1 y_1\, x_1^{-1} S\; RQ\; y_1^{-1}\, T$

$\longrightarrow \quad y_1 x_2^{-1}\, y_1^{-1}\, TP\;\; SRQ\; x_2$

$\longrightarrow \quad x_2\, y_1\, x_2^{-1}\, y_1^{-1}\; TPSRQ$

- so reduced to a certain # of adjacent aa and a certain # of $aba^{-1}b^{-1}$ instructions

Note: no remaining a, a^{-1} pairs without a crossed b, b^{-1}:

if 

any b edge in matched to a b or b^{-1} in same region ; same for below
(otherwise a belongs to another $aba^{-1}b^{-1}$)

but then $P \neq Q$ identified vertices

(know can reduce to single identif. vertex)

- final presentation:

each sequence $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ is a conn. sum $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g}$

and each seq. $a_1 a_1 \cdots a_h a_h$ is a conn. sum $\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{h}$

then use relation
$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 = \mathbb{T}^2 \# \mathbb{P}^2$ to simplify

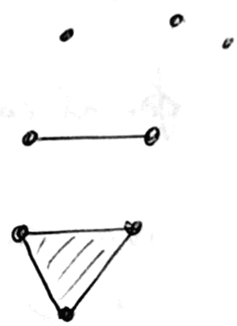
final thing needed:

every topological surface can be described by a gluing polygon w/ sides pairwise identified

every 2-dimensional topological surface admits a triangulation (finite since X compact)

Triangulations

* a set of Vertices V
* a set of 1-edges E
* a set of 2-simplexes (triangles) or faces F



s.t. $X = \bigcup_{f \in F} F$ $\quad \mathring{F}_i \cap \mathring{F}_j = \emptyset$

interiors of faces do not overlap

$F_i \cap F_j$ is a union of edges and vertices s.t.

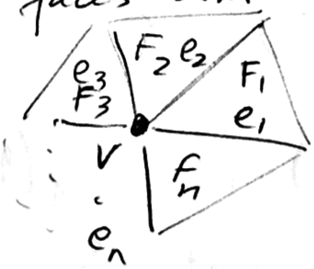
each $e \in E$ contained in exactly two faces F_i, F_j

for each $v \in V$

$E_v \subseteq E$ edges containing v

$F_v \subseteq F$ faces containing v

arranged cyclically around v



From a triangulation of X produce a polygon diagram by cutting

Last step to know classification is complete:

Show $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g} \cong \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g'}$ iff $g = g'$

and $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g} \# \mathbb{P}^2$ not homeom to any

and homeom to other only if $g = g'$

same for $\underbrace{\mathbb{T}^2 \# - \# \mathbb{T}^2}_{g} \# K$

each final case of classif. not homeom. to another one

Need an invariant (homeomorphism invariant)
that can be computed for each case and not same

1) orientability is homeom. invariant

2) Euler characteristic

$$\chi(X) = \#V - \#E + \#F$$

Claim: $\chi(X)$ does not depend on the triangulation only on X

for $\underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g} = X$ $\quad \chi(X) = 2 - 2g$

for S^2 $\quad \chi(S^2) = 2$

for $\mathbb{P}^2$ $\quad \chi(\mathbb{P}^2) = 1$ & $\chi(\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{h}) = 2 - h$

so $\chi(\mathbb{T}^2) = \chi(\text{Klein}) = 0$ but one orientable one not