Filters

X a set $\mathcal{P}(X) \supseteq \mathcal{F}$ filter if

(i) if $A, B \in \mathcal{F}$ $\exists C \in \mathcal{F}$ $C \subseteq A \cap B$ ($\mathcal{F}$ is "downward directed")

(ii) $\mathcal{F} \neq \emptyset$

(iii) $A \in \mathcal{F}$ $A \subseteq B \Rightarrow B \in \mathcal{F}$ ($\mathcal{F}$ is "upward closed")

(iv) $\exists A \in \mathcal{P}(X)$ $A \notin \mathcal{F}$ (proper) i.e. proper iff $\emptyset \notin \mathcal{F}$

(i) & (iii) $\Rightarrow$ $\mathcal{F}$ closed under finite intersections

A Construction ~~Characterization~~ of filters

set $2 = (\{0,1\}, \leq)$ $0 \leq 1$

poset of 2 pts

w/ operation defined by

$0 \wedge 0 = 0$ $0 \wedge 1 = 0$

$1 \wedge 1 = 1$ $1 \wedge 0 = 0$

(logical AND operation)

$\mathcal{P}(X)$

w/ operation

$A \cap B$

(also logical AND operation)

also poset w/ inclusion

consider functions $f : \mathcal{P}(X) \to 2$ monotone w/ resp to poset structures

and compatible w/ AND operation

$$f(A \cap B) = f(A) \wedge f(B)$$

$F = f^{-1}(1)$ is a filter

Example:

$\mathcal{F} = \{A \subseteq X \mid X \setminus A \text{ is finite}\}$ is a filter

(the Fréchet filter)

$(X, \mathcal{T}_X)$ topological space

$\mathcal{F} \subseteq \mathcal{P}(X)$ filter

$x \in X$ $\quad \mathcal{T}_x \subseteq \mathcal{T}_X$ open neighborhoods of x

$$\mathcal{T}_x = \{U \in \mathcal{T}_X \mid x \in U\}$$

the filter $\mathcal{F}$ <u>converges</u> to $x \in X$ iff $\mathcal{T}_x \subseteq \mathcal{F}$

<u>Notation</u>: $\mathcal{F} \longrightarrow x$ or $\mathcal{F} \xrightarrow[\mathcal{T}_X]{} x$

$(X, \mathcal{T}_X)$ is Hausdorff iff

limits of convergent proper filters are unique

- if X hausdorff and $\mathcal{F}$ proper filter with $\mathcal{F} \to x$ & $\mathcal{F} \to y$ with $x \neq y$

then since $x \neq y$ in Hausdorff space $\exists\, U, V \in \mathcal{T}_X$

$U \cap V = \emptyset \quad x \in U$ & $y \in V$

but $\mathcal{F}$ proper filter:
- by convergence $U \in \mathcal{F}$ & $V \in \mathcal{F}$
- filter $U \cap V = \emptyset \in \mathcal{F}$
- proper $\emptyset \notin \mathcal{F}$

<u>pushforward of a filter</u> $\mathcal{F} \subseteq \mathcal{P}(X)$ filter
$f: X \to Y$ function

$$f_* \mathcal{F} = \{ B \subseteq Y \mid \exists A \in \mathcal{F} \text{ s.t. } f(A) \subseteq B \}$$

(note that set of images $f(A)$ by itself need not be a filter)

<u>Continuity described through filters</u>

$f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous iff

for every filter $\mathcal{F} \subseteq \mathcal{P}(X)$ with $\mathcal{F} \to x$ $\Rightarrow$ $f_* \mathcal{F} \to f(x)$

this relates open sets def of topological continuity w/ pointwise def of <u>continuity at x</u>

- $\mathcal{F}$ filter $\mathcal{F} \to x$ & $f: X \to Y$ continuous
  then want to check neighborhoods of $f(x)$ are in pushforward filter
  $\mathcal{T}_{f(x)} \subseteq f_* \mathcal{F}$ (this gives convergence $f_* \mathcal{F} \to f(x)$)

  so want: $\forall\, \mathcal{U} \in \mathcal{T}_{f(x)}$ $(\mathcal{U} \in \mathcal{T}_Y \;\; f(x) \in \mathcal{U})$
  $\exists\, A \in \mathcal{F}$ w/ $f(A) \subseteq \mathcal{U}$
  take $A = f^{-1}(\mathcal{U})$ f contin. so $f^{-1}(\mathcal{U}) \in \mathcal{T}_X$ containing x
  so $f^{-1}(\mathcal{U}) \in \underbrace{\mathcal{T}_x \subseteq \mathcal{F}}_{\text{by conv } \mathcal{F} \to x}$

- Conversely $\mathcal{F} \to x \Rightarrow f_* \mathcal{F} \to f(x)$
  this for any filter
  take $\mathcal{F}$ = <u>filter generated by</u> $\mathcal{T}_x$ = all not nec. open neighb. $\mathcal{F}_x$
  then $f_* \mathcal{F} \to f(x)$ means $\mathcal{T}_{f(x)} \subseteq f_* \mathcal{F} \Rightarrow$ $\forall\, \mathcal{U} \in \mathcal{T}_{f(x)}$ $\exists\, A \in \mathcal{F}_x$ $f(A) \subseteq \mathcal{U}$
  $\Rightarrow f$ continuous

Tychonoff theorem about compactness of products

$X = \prod_{\alpha \in I} X_\alpha$ if $(X_\alpha, \mathcal{T}_\alpha)$ compact for all $\alpha \in I$ then X is compact in product topology

proof via ultrafilters

Ultrafilter = a proper filter $\mathcal{F} \subset \mathcal{P}(X)$ that is maximal i.e. not properly contained in any other proper filter

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Equivalent condition

$\mathcal{F} \subset \mathcal{P}(X)$ ultrafilter iff $\forall A \in \mathcal{P}(X)$

$$A \notin \mathcal{F} \iff \exists B \in \mathcal{F} \text{ s.t. } A \cap B = \emptyset \qquad (*)$$

Pf: $\mathcal{F}$ ultrafilter : then $A \notin \mathcal{F}$ iff filter generated by A and $\mathcal{F}$ is not proper i.e. is $\mathcal{P}(X)$

so it must contain $\emptyset$ i.e. there has to be a set $B \in \mathcal{F}$ s.t. $\underbrace{A \cap B}_{\text{in filter gen by } A \,\&\, \mathcal{F}} = \emptyset$

if $\mathcal{F}$ filter satisfying $(*)$ and $\mathcal{F}'$ another filter w/ $\mathcal{F} \subset \mathcal{F}'$

$\exists A \in \mathcal{F}'$ $A \notin \mathcal{F}$

so $\exists B \in \mathcal{F}$ $A \cap B = \emptyset \Rightarrow \mathcal{F}' = \mathcal{P}(X)$

so $\mathcal{F}$ is maximal (i.e. ultrafilter)

Example:

$x \in X$ principal filter of x

$= \mathcal{F}_x := \{A \subseteq X \mid x \in A\}$ is an ultrafilter

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A filter $\mathcal{F} \subseteq P(X)$ is prime iff
it is proper and $\forall A, B \in P(X)$

$$A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \text{ OR } B \in \mathcal{F}$$

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ultrafilter $\Leftrightarrow$ prime

$\mathcal{F}$ ultrafilter: if not prime $\exists A, B$ $\quad A \cup B \in \mathcal{F}$
but neither in $\mathcal{F}$

since ultrafilter $\exists A', B' \in \mathcal{F}$

$A \cap A' = \emptyset \qquad B \cap B' = \emptyset$

contradicts this

So $(A \cup B) \cap (A' \cup B') = \emptyset$

but this $\Rightarrow (A \cup B) \notin \mathcal{F}$ because $\mathcal{F}$ proper

$\mathcal{F}$ prime: if not ultrafilter $\exists \mathcal{F}'$ proper w/ $\mathcal{F} \subset \mathcal{F}'$ strictly

$\exists A \in \mathcal{F}' \quad A \notin \mathcal{F}$

$\Downarrow$

contradicts this

$X \setminus A \notin \mathcal{F}$ (as otherwise
if $X \setminus A \in \mathcal{F}$ also
$X \setminus A \in \mathcal{F}'$ & $\mathcal{F}'$ not proper)

but since $\mathcal{F}$ prime and $X = A \cup (X \setminus A) \in \mathcal{F}$
would have ~~either~~ A or $X \setminus A \in \mathcal{F}$

Zorn's lemma: if every chain in a nonempty poset $\mathbb{P}$ has an upper bound then $\mathbb{P}$ has a maximal element

Ultrafilter existence:

every proper filter $\mathcal{F} \subset P(X)$ is contained in an ultrafilter

from Zorn: any set of proper filters $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ bounded above by filter $\mathcal{F}^{sup}$ generated by finite intersections of sets in the $\mathcal{F}_\alpha$'s

if chain of filters $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ $\alpha \leq \beta$ in I

thus $\mathcal{F}^{sup}$ bound is proper also

Start w/ a proper filter $\mathcal{F}$ consider chains of proper filters containing $\mathcal{F}$ $\{\mathcal{F}'_\alpha\}_{\alpha \in I}$ with $\mathcal{F}'_\alpha \supseteq \mathcal{F}$

then $\mathcal{F}^{sup}$ of these chains proper upper bounds

$\Rightarrow$ by Zorn's $\exists$ max element $\mathcal{F}' \supseteq \mathcal{F}$ of $\mathbb{P}$ = poset of proper $\mathcal{F}' \supseteq \mathcal{F}$

$\mathcal{F}'$ proper

$(X, \mathcal{T}_X)$ is compact iff every ultrafilter converges

Pf: ultrafilter $\Leftrightarrow$ prime

- suppose $\mathcal{F}$ prime filter not converging to any $x \in X$

equivalently for all $x \in X$ $\quad \exists\, U_x \in \mathcal{T}_x \setminus \mathcal{F}$

the set $\{U_x\}_{x \in X}$ open covering of X

X compact $\Rightarrow$ $\exists$ finite subcovering

$U_{x_1} \cup \cdots \cup U_{x_n} = X$

but $\mathcal{F}$ prime so since $X \in \mathcal{F}$

some $U_{x_i} \in \mathcal{F}$ contradicts

- if X not compact $\quad \exists\, \mathcal{V}$ collection of closed sets $= \{V_\alpha\}_{\alpha \in I}$

$\bigcap_{\alpha \in I} V_\alpha = \emptyset$

but all fin intersections $V_{\alpha_1} \cap \cdots \cap V_{\alpha_N} \neq \emptyset$

$\forall x \in X \quad \exists\, \alpha(x)$ s.t. $V_{\alpha(x)} \not\ni x$

$\exists\, \mathcal{F}$ ultrafilter s.t. $\mathcal{V} \subseteq \mathcal{F}$ $\quad$ (by existence of ultrafilters) and using filter gen by $\mathcal{V}$

this $\mathcal{F}$ cannot converge anywhere

as if $\mathcal{F} \to x$ for some $x \in X$

then $\mathcal{T}_x \subseteq \mathcal{F}$ and $\mathcal{T}_x \ni V^C_{\alpha(x)}$

so have both $V_{\alpha(x)}$ and $V^C_{\alpha(x)} \in \mathcal{F}$ with $V_{\alpha(x)} \cap V^C_{\alpha(x)} = \emptyset$

so $\mathcal{F}$ cannot be proper contradicting

Tychonoff theorem via ultrafilters

$\{X_\alpha\}_{\alpha \in I}$ $\quad (X_\alpha, \mathcal{T}_\alpha)$ compact

$X = \prod_\alpha X_\alpha$ w/ prod. topology

$\mathcal{F}$ an ultrafilter in X

pushforward of an ultrafilter is also an ultrafilter

so $(\pi_\alpha)_* \mathcal{F} =: \mathcal{F}_\alpha$ ultrafilter in X_α

since X_α compact $\mathcal{F}_\alpha \longrightarrow x_\alpha$ for some $x_\alpha \in X_\alpha$

so for all $U_\alpha \in \mathcal{T}_{x_\alpha} \subseteq \mathcal{T}_{X_\alpha}$ $\quad \exists\, B \in \mathcal{F}$

with $\pi_\alpha(B) \subseteq U_\alpha$

so $B \subseteq \pi_\alpha^{-1}(U_\alpha)$

in $\mathcal{F}$ so $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}$

every U open neighborhood of $x = (x_\alpha) \in X$ in prod. topology

is a union of fin. intersections of sets $\pi_\alpha^{-1}(U_\alpha)$ so also in $\mathcal{F}$

$\Rightarrow \mathcal{F} \longrightarrow x = (x_\alpha)$

(X, T_X) is compact and Hausdorff
iff every ultrafilter has exactly one limit point

X $\qquad$ $\beta X = \cancel{\{}$ set of all ultrafilters $\mathcal{F}$ on X

functor β : Sets $\longrightarrow$ Sets

$\beta : X \longmapsto \beta X$ & for $f : X \to Y$

$$\beta(f) : \beta X \to \beta Y$$
$$\mathcal{F} \longmapsto f_* \mathcal{F}$$

(again use pushforward maps ultrafilters to ultrafilters)

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if (X, T_X) compact and Hausdorff
also can assign to a filter $\mathcal{F}$ its unique limit pt $x \in X$ w/ $\mathcal{F} \to x$

So obtain a map of sets $\quad \beta X \longrightarrow X$

$$\mathcal{F} \longmapsto \lim \mathcal{F}$$

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category $\mathcal{C}$ a <u>monad</u> in $\mathcal{C}$

is given by

- an endofunctor $T: \mathcal{C} \to \mathcal{C}$
- a natural transformation

$$\eta : \underset{\text{id. functor}}{1_{\mathcal{C}}} \to T \quad \text{(unit of the monad)}$$

- a multiplication: natural transformation

$$\mu : T \circ T \to T$$

w/ property that these diagrams commute

$$\begin{tikzcd}
T\circ T\circ T \arrow[r, "T\circ\mu"] \arrow[d, "\mu\circ T"'] & T\circ T \arrow[d, "\mu"] \\
T\circ T \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta\circ T"] \arrow[dr, "1_T"'] & T\circ T \arrow[d, "\mu"] & T \arrow[l, "T\circ\eta"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}$$

- modelled on the algebraic structure of "monoid"

<u>Adjunction of functors</u> $\mathcal{C}, \mathcal{D}$ categories

$L: \mathcal{C} \to \mathcal{D}$ $R: \mathcal{D} \to \mathcal{C}$ functors adjunct pair iff

$$\mathrm{Mor}_{\mathcal{D}}(L(X), Y) \underset{\text{sets}}{\cong} \mathrm{Mor}_{\mathcal{C}}(X, R(Y))$$

L = left adjoint R = right adjoint

Notation $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$ to indicate F is left-adj to G

<u>Example</u> $U: \underline{Top} \longrightarrow \underline{Sets}$ forgetful functor

$$(X, \mathcal{T}_X) \xmapsto{U} X$$

$$f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \longmapsto f: X \to Y \quad \text{as map of sets}$$
contin

U has both a left and a right adjoint

<u>Left adjoint</u>: $S \in \underline{Sets}$ want to assign to this a topological space $D(S)$ s.t.

continuous maps from $D(S)$ to any other $(Y, \mathcal{T}_Y)$ are the same as maps of sets from S to Y

$$\mathrm{Mor}_{Top}(D(S), (Y, \mathcal{T}_Y)) \cong \mathrm{Mor}_{Sets}(S, Y)$$

so want topology on $D(S)$ so that any $f: S \to Y$ for any topology $\mathcal{T}_Y$ is continuous

need to use $D(S) = (S, \mathcal{P}(S))$ discrete topology

so $D: \underline{Sets} \to \underline{Top}$ considers set S as top space w/ discrete top.

right adjoint: want $I(S)$ topological space assoc to set S s.t. continuous maps from $(X, \mathcal{T}_X)$ to $I(S)$ are same as maps of sets from X to S

$$\mathrm{Mor}_{Top}((X, \mathcal{T}_X), I(S)) \cong \mathrm{Mor}_{Sets}(X, S)$$

need to take $I(S) = (S, \{\emptyset, S\})$
indiscrete topology

only one when all function of sets $f: X \to S$ are continuous in whatever topology $\mathcal{T}_X$ on X

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad \text{s.t.}$$

$$\mathrm{Mor}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Mor}_{\mathcal{C}}(X, G(Y))$$

have "unit" and "counit" of the adjunction

natural transformations

$$\eta: 1_{\mathcal{C}} \to G \circ F \qquad \varepsilon: F \circ G \to 1_{\mathcal{D}}$$

with identities

$$\begin{tikzcd} F \arrow[r, "F\circ\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

case of $\eta : 1_{\mathcal{C}} \to G \circ F$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & G(F(X)) \arrow[d, "GF(f)"] \\ X' \arrow[r, "\eta_{X'}"] & GF(X') \end{tikzcd}$$

$$\underbrace{\begin{array}{c} Mor_{\mathcal{C}}(X, GF(X)) \\ \cong \\ Mor_{\mathcal{D}}(F(X), F(X)) \end{array}}_{\text{this always contains } 1_{F(X)} \text{ id morphism}}$$

so this corresp. to a morphism in $Mor_{\mathcal{C}}(X, GF(X))$ call that η_X

other case of ε similar

Adjunctions of functors determine monads

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

$\eta : 1_{\mathcal{C}} \to U \circ F$

$\varepsilon : F \circ U \to 1_{\mathcal{D}}$

- endofunctor $T = U \circ F$
- unit $\eta : 1_{\mathcal{C}} \to T = U \circ F$
- multiplication $\mu : T \circ T \to T$

$\mu = U \varepsilon F : UFUF \to UF$

(the three items above: monad in $\mathcal{C}$)

Role of $\mathcal{D}$? Algebras over the monad in $\mathcal{C}$

(w/ some conditions for adjunction to be "monadic") functor $\beta : \underline{\text{Sets}} \longrightarrow \underline{\text{Sets}}$

$X \longmapsto \beta X$ set of ultrafilters on X

is in fact composite of two adjunctions

$$\begin{tikzcd} \text{CHaus} \arrow[r, shift right, hook, "j \text{ incl.}"'] \arrow[r, phantom, "\perp"] & \underline{\text{Top}} \arrow[l, shift right, "\tilde\beta"'] \arrow[r, shift right, "U \text{ (forgetful)}"'] \arrow[r, phantom, "\perp"] & \underline{\text{Sets}} \arrow[l, shift right, "D \text{ discrete top}"'] \end{tikzcd}$$

$\tilde\beta$ = Stone-Čech compactification

$\beta = U \circ j \circ \tilde\beta \circ D$ so monad on Sets

CHaus compact hausdorff topological spaces are algebras over the monad β