Topology 9

local compactness: $\forall x \in X \ \exists K \subseteq X$ compact & $\mathcal{U}(x) \in \mathcal{T}$ st. $x \in \mathcal{U}(x) \subset K$ (at every pt. $\exists$ comp. set containing a neighb.)

$(X, \mathcal{T})$ <u>normal</u> if for any A, B disjoint & closed in X
$\exists$ disjoint open neighb $U(A)$, $V(B)$ in X
$A \subseteq U(A)$ $\quad B \subseteq V(B)$ $\quad U(A) \cap V(B) = \emptyset$

(normal $\Rightarrow$ Hausdorff)
(compact + Hausdorff $\Rightarrow$ normal)

(X, d) $\quad \mathcal{T} = \mathcal{T}_d$ is normal

[necessary condition for metrizability]

A, B disjoint closed in X

$a \in A$ choose $\varepsilon_a > 0$ s.t.
$B_d(a, \varepsilon_a) \cap B = \emptyset$
(i.e. $B_d(a, \varepsilon_a) \subseteq B^c$ open)

similarly for $b \in B$ $\varepsilon_b > 0$
$B_d(b, \varepsilon_b) \cap A = \emptyset$

$$U(A) = \bigcup_{a \in A} B_d(a, \varepsilon_a/2) \supseteq A \qquad V(B) = \bigcup_{b \in B} B_d(b, \varepsilon_b/2) \supseteq B$$

if $z \in U(A) \cap V(B)$ then $z \in B_d(a, \varepsilon_a/2) \cap B_d(b, \varepsilon_b/2)$
for some $a \in A$, $b \in B$

so $d(a,b) < \frac{\varepsilon_a}{2} + \frac{\varepsilon_b}{2}$ by triangle ineq
$d(a,z) + d(z,b)$

but then if $\varepsilon_a \leq \varepsilon_b$ $\quad B(b, \varepsilon_b) \ni a$ because $d(a,b) < \varepsilon_b$
if $\varepsilon_a \geq \varepsilon_b$ $\quad B(a, \varepsilon_a) \ni b$ " $d(a,b) < \varepsilon_a$

so contradicts $B(a, \varepsilon_a) \cap B = \emptyset$ & $B(b, \varepsilon_b) \cap A = \emptyset$

Urysohn Lemma $(X, \mathcal{T})$ normal

A, B disjoint closed subsets in X

can be separated by continuous functions

$I = [a,b] \subseteq \mathbb{R}$ interval (Eucl. top.)

$\exists \; f: X \to [a,b]$ continuous s.t.

$f(x) = a \quad \forall x \in A \quad \& \quad f(x) = b \quad \forall x \in B$

Pf: assume $I = [0,1]$ $\qquad I \cap \mathbb{Q} = P$ rational pts of I (dense subset, countable)

$\mathcal{U}_p \subseteq X$ open set $\quad p \in I \cap \mathbb{Q}$

s.t. for $p < q$ $\quad \overline{\mathcal{U}}_p \subseteq \mathcal{U}_q$

construct inductively ($I \cap \mathbb{Q}$ countable)

enumerate pts of $I \cap \mathbb{Q}$ $\{p_n\}$ $\quad$ say with $p_1 = 1$, $p_2 = 0$ $\cdots p_n \cdots$

$\mathcal{U}_1 = X \setminus B \qquad A \subseteq \mathcal{U}_1$ closed subset in normal space

$\exists$ open set $U(A) \supseteq A$ s.t. $\overline{U(A)} \subset \mathcal{U}_1$

$U(A) = \mathcal{U}_0 \qquad \mathcal{U}_0 \subseteq \mathcal{U}_1$

suppose constructed first n $\mathcal{U}_p$ sets w/ $\overline{\mathcal{U}}_p \subseteq \mathcal{U}_q$ for $p < q$

then r next rational in sequence

$\exists \; p, q$ in first rationals $\quad p < r < q$ (in ordering of $\mathbb{R}$)

these satisfy $\overline{\mathcal{U}}_p \subset \mathcal{U}_q$ $\qquad$ by X normal: $\exists \; \mathcal{U}(\overline{\mathcal{U}}_p)$

s.t. $\overline{\mathcal{U}}_p \subset \mathcal{U}(\overline{\mathcal{U}}_p)$ and $\overline{\mathcal{U}(\overline{\mathcal{U}}_p)} \subset \mathcal{U}_q$

take $\mathcal{U}_r = \mathcal{U}(\overline{\mathcal{U}}_p)$ as next set now with these open sets $\{U_p\}_{p\in I\cap\mathbb{Q}}$

extend to $U_p=\emptyset$ if $p\in\mathbb{Q}\cap(-\infty,0)$

$U_p = X$ if $p\in\mathbb{Q}\cap(1,\infty)$

take $x\in X$ $\qquad \mathbb{Q}(x)=\{p\in\mathbb{Q} \mid x\in U_p\}$ bounded below (by $p=0$)

~~least~~ greatest lower bound is ≥ 0

take $f(x)=\inf\mathbb{Q}(x)$

if $x\in A$ then $x\in U_p$ all $p\geq 0$ $\Rightarrow f(x)=0$

if $x\in B$ then $x\notin U_p$ all $p\leq 1$ $\Rightarrow f(x)=1$

need to check $f(x)$ is continuous

- if $x\in\overline{U}_r$ then $x\in U_s$ for all $s>r$ $\Rightarrow \overset{f(x)}{\overset{\shortparallel}{\inf\mathbb{Q}(x)}}\leq r$
- if $x\notin U_r \Rightarrow x\notin U_s$ for $s<r$ $\Rightarrow f(x)=\inf\mathbb{Q}(x)\geq r$

given $x_0\in X$ & $\underset{f(x_0)}{\underset{\ni}{(c,d)}}$ open in $\mathbb{R}$

find neighb. $U(x_0)$ with $f(U(x_0))\subset(c,d)$

$\exists\, p,q\in\mathbb{Q}$ s.t. $\quad c<p<f(x_0)<q<d$

$U=U_q\setminus\overline{U}_p$ open has right properties

- $x_0\in U$ $\qquad f(x_0)<q \Rightarrow x_0\in U_q$
  $\qquad f(x_0)>p \Rightarrow x_0\notin U_p$
- $f(U)\subset(c,d)$ because
  $x\in U \Rightarrow x\in U_q \Rightarrow f(x)\leq q$
  $\qquad x\notin\overline{U}_p \Rightarrow f(x)\geq p$

Urysohn metrization

$(X, \mathcal{T})$ regular space if $\forall x, B$ (closed) $x \notin B$
$\exists\, U(x), V(B)$ $\quad U(x) \cap V(B) = \emptyset$

(normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff)

If $(X, \mathcal{T})$ is regular & has a countable basis $\mathcal{B}$
then it is metrizable

show embedded in metrizable space

$\exists$ countable $\{f_n\}$ $\quad f_n : X \to [0,1]$ s.t.

given $x_0 \in X$ and $\mathcal{U}(x_0)$ $\quad \exists\, n$ with $f_n(x_0) > 0$

$f_n |_{\mathcal{U}(x_0)^c} \equiv 0$

because of Urysohn lemma: $x_0 \in \mathcal{U}$ $\quad \exists\, f_{x_0,\mathcal{U}}(x_0) = 1$ $\quad f_{x_0,\mathcal{U}}|_{\mathcal{U}^c} \equiv 0$

but $\{f_{x_0,\mathcal{U}}\}_{x_0 \in X}$ can be uncountable

$\{B_n\} = \mathcal{B}$ countable basis $\quad (n,m)$ pairs w/
$\overline{B}_n \subset B_m$

$\exists\, g_{n,m} : X \to [0,1]$ $\quad g_{n,m}|_{\overline{B}_n} \equiv 1$ $\quad g_{n,m}|_{X \setminus B_m} \equiv 0$

given $x_0 \in X$ and $\mathcal{U}(x_0)$ $\quad x_0 \in B_n$ some n and $\overline{B}_n \subset B_m$ some m

then $g_{n,m}$ has right prop.

relabel so single index $\{f_n\}$ since countable set then given these $\{f_n\}$

$F : X \longrightarrow [0,1]^{N}$ product topology

$x \longmapsto (f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots)$

- Continuous because each f_n is
- injective $x \neq y$ $\exists\, n$ $f_n(x) > 0$ $f_n(y) = 0$
  $\Rightarrow F(x) \neq F(y)$

- surjective to image $F(X)$

- open map: $U \subseteq X$ open $F(U) \subseteq [0,1]^{N}$

$F : X \to F(X)$

$z_0 = F(x_0)$

$f_N(x_0) > 0$ $f_N(X \setminus U) = \{0\}$

$V = \pi_N^{-1}((0,\infty))$ $W = V \cap F(X)$ open in ind. top.

$z_0 \in W$: $\pi_N(z) = \pi_N(F(x_0)) = f_n(x_0) > 0$

$W \subset F(U)$ $z \in W$ is $z = F(x)$ for some x
w/ $\pi_N(z) = \pi_N(F(x)) = f_N(x) \in (0,\infty)$

$f_N|_{U^c} \equiv 0$ So if $x \in U$

so $z \in F(U)$

So $\forall z \in F(U)$
$\exists\, W$ open $z \in W \subseteq F(U)$ $\Rightarrow F(U)$ open

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$[0,1]^{\mathbb{N}}$ is metrizable $d(x,y) = \sup_i \dfrac{|x_i - y_i|}{i}$

$x = (x_i)_{i=1}^{\infty}$

$x_i \in [0,1]$ is a metric on $[0,1]^{\mathbb{N}}$

this metric induces product topology

- $\mathcal{U}$ open in $\mathcal{T}_d$ $\quad \exists\, \mathcal{V}$ open in prod top
  $\forall x \in \mathcal{U}$ $\quad$ s.t. $x \in \mathcal{V} \subseteq \mathcal{U}$

$B_d(x, \varepsilon)$ in $\mathcal{U}$ $\quad N >> 0$ s.t. $\frac{1}{N} < \varepsilon$

$\mathcal{V} = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times I \times I \times \cdots$

$\mathcal{V} \subseteq B_d(x, \varepsilon)$ since $\forall y \in I^N$

$$\frac{|x_i - y_i|}{i} \le \frac{1}{N} \quad \text{for } i \ge N$$

$$\Rightarrow \quad d(x,y) \le \max\left\{ |x_1 - y_1|, \frac{|x_2 - y_2|}{2}, \ldots, \frac{|x_N - x_N|}{N}, \frac{1}{N} \right\}$$

if $y \in \mathcal{V}$ $\quad d(x,y) < \varepsilon \quad \Rightarrow \mathcal{V} \subseteq B_d(x, \varepsilon)$

- $\mathcal{U} = \mathcal{U}_{\alpha_1} \times \cdots \times \mathcal{U}_{\alpha_n} \times \prod_{\alpha \notin \{\alpha_1, \ldots, \alpha_n\}} I$ $\quad$ neighb. in prod. top

$\mathcal{U}_{\alpha_i} \subseteq I$ open

$x \in \mathcal{U}$ $\quad \exists\, \mathcal{V}$ open in metric topology
s.t. $x \in \mathcal{V} \subseteq \mathcal{U}$

pick $(x_{\alpha_i} - \varepsilon_{\alpha_i}, x_{\alpha_i} + \varepsilon_{\alpha_i}) \subset I$ $\quad$ s.t. $(x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subseteq \mathcal{U}_{\alpha_i} \subseteq I$

$$\varepsilon = \min\left\{ \frac{\varepsilon_{\alpha_i}}{\alpha_i} \right\}_{i=1}^{n}$$

$\Rightarrow x \in B_d(x, \varepsilon) \subseteq \mathcal{U}$

since $y \in B_d(x, \varepsilon)$: for all i

$$\frac{|x_i - y_i|}{i} \le d(x,y) < \varepsilon$$

$\varepsilon \le \frac{\varepsilon_{\alpha_i}}{i} \quad \Rightarrow \quad |x_{\alpha_i} - y_{\alpha_i}| < \varepsilon_{\alpha_i} \quad \Rightarrow \quad y \in \mathcal{U}_{\alpha_1} \times \cdots \times \mathcal{U}_{\alpha_n} \times \prod_{\alpha} I$

Topology $\boxed{9b}$

Separation axioms:

T_0 $\forall\, x \neq y \in X$ $\exists\, U \in \mathcal{T}$ with either $x \in U$ & $y \notin U$ or $x \notin U$ & $y \in U$

T_1 $\forall\, x \neq y \in U$ $\exists\, V, U \in \mathcal{T}$ $x \in U$ & $y \notin U$, $x \notin V$ & $y \in V$

T_2 Hausdorff $\forall\, x \neq y \in U$ $\exists\, U, V \in \mathcal{T}$ $x \in U$ $y \in V$ $U \cap V = \emptyset$

T_3 regular & T_0 A closed, $x \notin A$ $\exists\, U, V \in \mathcal{T}$ $x \in U$ $A \subseteq V$ $U \cap V = \emptyset$

T_4 normal & T_1 A, B closed $A \cap B = \emptyset$ $\exists\, U, V$ $A \subseteq U$ $B \subseteq V$ $U \cap V = \emptyset$