$(X, \mathcal{T})$ compact $\qquad f: X \to \mathbb{R}$ (Eucl. top) continuous

then $\exists\, x_0, x_1 \in X$ st. $f(x_0) \le f(x) \le f(x_1) \quad \forall x \in X$
(f has a min & a max on X)

In fact: if $Y = f(X) \subseteq \mathbb{R}$ has no largest element

then $\forall y \in Y \quad (-\infty, y) \cap Y$ open covering of Y in induced top.

but X compact $\Rightarrow Y = f(X)$ compact

so $\exists$ finite subcovering $((-\infty, y_1)^{\cap Y} \cup \dots \cup (-\infty, y_n)^{\cap Y} = Y$

take $y = \max\{y_1, \dots, y_n\} \quad y \notin (-\infty, y_i)$

in Y but in none of $\quad$ so cannot be

$\Rightarrow Y$ must have a largest element

similar for smallest

(X, d) metric $\quad \mathcal{T}_d$ metric top.

$Y \subseteq X$ nonempty $\qquad d(x, Y) = \inf\{d(x,y) \mid y \in Y\}$
$x \in X$

$d(x, Y)$ continuous function of $x \in X$ w/ real values

since $d(x_1, Y) - d(x_2, Y) \le d(x_1, x_2)$

$(\ d(x_1, y) \le d(x_1, x_2) + d(x_2, y)$ then take inf on $y\ )$

Lebesgue number of a covering

$\mathcal{U} = \{U_\alpha\}$ open covering of (X, d) <u>compact</u>

assume X itself not an open set of the covering

$\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ finite subcovering

$C_{\alpha_i} = X \setminus U_{\alpha_i}$ complement closed sets

$f : X \to \mathbb{R}$ defined as $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_{\alpha_i})$

average distance from these closed sets

given $x \in X$ $\exists\, U_{\alpha_i}$ s.t. $x \in U_{\alpha_i}$ hence $\exists\, \varepsilon > 0$ $B_d(x, \varepsilon) \subseteq U_{\alpha_i}$

$\Rightarrow d(x, C_{\alpha_i}) \geq \varepsilon$ hence $f(x) > 0$ for all $x \in X$

f continuous (since $d(x, C_{\alpha_i})$ is) hence has min & max on X

$\delta := \min_{x \in X} f(x)$ Lebesgue number of $\mathcal{U}$

is such that for each subset $Y \subseteq X$ with

$\mathrm{diam}(Y) < \delta$ $\exists\, U_\alpha \in \mathcal{U}$ $Y \subseteq U_\alpha$

(whenever a subset has size less than δ it is contained in one of the open sets in the covering)

indeed take $Y \subseteq X$ w/ $\mathrm{diam}(Y) = \sup\{d(x,y) \mid x, y \in Y\} < \delta$

choose a pt $y \in Y$ diam $< \delta$ implies $Y \subseteq B_d(y, \delta)$

but $\delta \leq f(y) \leq d(y, C_\alpha)$ hence $Y \subseteq B_d(y, \delta) \subseteq U_\alpha$

largest among $d(y, C_{\alpha_1}), \dots, d(y, C_{\alpha_n})$

<u>Uniform continuity</u> $f:(X,d_X)\to(Y,d_Y)$ continuous

and (X,d_X) compact

then f is <u>uniformly continuous</u> i.e.

$\forall \varepsilon>0 \ \exists\delta>0$ s.t. $d_X(x,y)<\delta \Rightarrow d_Y(f(x),f(y))<\varepsilon$

(indep of $x,y\in X$)

from previous Lebesgue number fact:

given $\varepsilon>0$ covering $\mathcal{U}_Y$ of Y by balls $\{B_{d_Y}(y,\frac{\varepsilon}{2})\}_{y\in Y}=\mathcal{U}_Y$

$\rightsquigarrow$ covering of X by $\{f^{-1}(B_{d_Y}(y,\frac{\varepsilon}{2}))\}=\mathcal{U}_X$

$\delta=$ Lebesgue # of $\mathcal{U}_X$

then if $d_X(x_1,x_2)<\delta$ $\{x_1,x_2\}\subseteq X$ has diam $<\delta$

$\Rightarrow \{f(x_1),f(x_2)\}\subseteq Y$ contained $\{f(x_1),f(x_2)\}\subseteq B_{d_Y}(y,\frac{\varepsilon}{2})$ for some $y\in Y$

then by triangle ineq.

$$d(f(x_1),f(x_2))\le d(f(x_1),y)+d(y,f(x_2))<\varepsilon$$

- $x\in X$ isolated pt. if $\{x\}$ open in $\mathcal{T}_X$

- X compact, Hausdorff, non empty
  no isolated pts $\Rightarrow$ uncountable

equiv. cond. to Hausdorff

$x\neq y$ $\exists\, U(x)$ ~~xxx~~ s.t. $y\notin\overline{U(x)}$

$\Rightarrow$ given U open and any $x\in X$ $\exists$ $V\subseteq U$ open $x\notin\overline{V}$

take $y\neq x\in U$ (if $x\in U$ need to know x not isolated pt) $\Rightarrow$ $\exists\, U(y)$ s.t. $x\notin\overline{U(y)}$; take $V=U(y)\cap U$

④ given $f: \mathbb{N} \to X$ show cannot be surjective ④

(hence X uncountable)

$x_n = f(n)$ by previous (w/ $U = X$) $\exists$ $V_1 \subseteq X$ $x_1 \notin \overline{V_1}$

etc. given V_{n-1} inductively construct V_n

with $V_n \subseteq V_{n-1}$ and $\overline{V_n} \not\ni x_n$ by previous

$\Rightarrow$ $\overline{V_1} \supseteq \overline{V_2} \supseteq \cdots$ closed subsets of X

Claim: X compact $\Rightarrow$ $\exists\, x \in \bigcap_n \overline{V_n}$

but $x \neq x_n$ because $x_n \notin \overline{V_n}$ so $\exists\, x \notin f(\mathbb{N})$ not surjective

proof of Claim:

equivalent characterization of compactness:

if $\bigcap_i F_i = \emptyset$ intersection of closed sets

then $\exists$ $F_{i_1}, \ldots F_{i_m}$ fin. many w/ $\bigcap_{k=1}^{m} F_{i_k} = \emptyset$

but for the $\overline{V_n}$ any finite subcollection

$\overline{V_{n_1}} \cap \cdots \cap \overline{V_{n_m}} \neq \emptyset$ so infinite inters. also $\neq \emptyset$

by construction: fix $y \in X$ and take smaller neighb's of y whose closure avoids x_n so all have pts in common finite steps

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Limit point compactness $(X, \mathcal{T})$

if every infinite subset $Y \subseteq X$ has an accumulation pt.

(in particular also all sequences have a convergent subsequence)

Compactness $\Rightarrow$ limit point compactness

show if $Y \subseteq X$ has no accumulation pts then it must be finite

$Y' = \emptyset \Rightarrow Y = \overline{Y}$ (since $\overline{Y} = Y \cup Y'$)

hence Y closed

for $y \in Y$ $\exists\, U(y)$ s.t. $U(y) \cap Y = \{y\}$

(if all neighb. intersect in pts other than y accum. pt.)

Covering of X by $\underset{\text{open}}{X \setminus Y}$ and $\{U(y)\}_{y \in Y}$

$\exists$ finite subcovering $\Rightarrow U(y_1) \dots U(y_n)$ cover Y

but each contains only one pt of Y

so Y finite

Sequential compactness: every sequence has a convergent subsequence

(X, d) metric $\mathcal{T}_d$

then equivalent

- X compact
- X limit point compact
- X sequentially compact

example for non metric space that reverse implication does not work:

$X = \{0,1\}$ top. $\{\emptyset, X\}$

$X \times \mathbb{N}$ w/ prod top.

not compact

$X \times \{n\}$ covering

no finite subcov.

limit point compact:

every nonempty subset has accum. pt.

$(x,n) \in X \times \{n\}$ accum. pt.

(x', n) $x' = \{0$ $x = 1$; $x = 0$

1) ⇒ 2) ⇒ 3) ok

need to show for (X, d) metric 3) ⇒ 1)

- if (X,d) seq. compact then argument about Lebesgue # applies

$\{U_\alpha\} = \mathcal{U}$ covering : suppose $\exists\, Y_n$ $\operatorname{diam}(Y_n) < \frac{1}{n}$ s.t.

$$Y_n \not\subseteq U_\alpha \text{ for any } \alpha$$

pick $x_n \in Y_n$ $\quad \exists$ convergent subseq $\{x_{n_k}\}$

$$\lim_{\to} x_{n_k} = x$$

$x \in U_\alpha$ for some α since U_α covering of X

so $B_d(x, \varepsilon) \subseteq U_\alpha$ when n_k suff. large $\frac{1}{n_k} < \frac{\varepsilon}{2}$

$$\Rightarrow Y_{n_k} \quad \operatorname{diam}(Y_{n_k}) < \frac{1}{n_k}$$

$$Y_{n_k} \subseteq B_d(x_{n_k}, \tfrac{\varepsilon}{2})$$

also for large enough n_k $\quad d(x_{n_k}, x) < \frac{\varepsilon}{2}$

$$\Rightarrow Y_{n_k} \underset{\substack{\uparrow \\ \text{triangle ineq.}}}{\subseteq} B_d(x, \varepsilon) \subseteq U_\alpha \quad \text{contradiction}$$

so $\exists\, \delta > 0$ if $\operatorname{diam}(Y) \leq \delta \Rightarrow Y \subseteq U_\alpha$ some α

- if (X,d) seq. compact for any given $\varepsilon > 0$
  $\exists$ covering finite of X by ε-balls

if not then ~~$\exists x_n$ s.t. $\{B(x_n, \varepsilon)\}$~~ $\quad \exists\, x_n$ sequence s.t.
(X cannot be covered by fin many ε-balls)

$x_1 \in X \quad B(x_1, \varepsilon)$

$x_2 \in X \setminus B(x_1, \varepsilon)$

$\vdots$

$x_n \in X \setminus \bigcup_{i=1}^{m} B(x_i, \varepsilon)$ etc.

$\Rightarrow d(x_n, x_{n+1}) \geq \varepsilon$ but then cannot have any conv. subseq.

then X compact : $\mathcal{U} = \{U_\alpha\}$ covering, δ = Lebesgue # $\mathcal{U}$ $\varepsilon = \delta/3$, $\exists$ finite covering by ε-balls diam $\leq \frac{2\varepsilon}{3} < \delta \Rightarrow$ each in U_α some α : fin many U_α suffice