Topology $\boxed{7}$ (1)

$(X, \mathcal{T})$ Topological space

Hausdorff (T_2 separation axiom)

1) if given any two $x \neq y \in X$

$\exists\, U, V \in \mathcal{T}$ $\quad x \in U \quad y \in V$

and $U \cap V = \emptyset$

(note: unlike connectedness doesn't require $U \cup V$ is everything)

Equivalently: 2) $\forall x \in X$ and $\forall y \neq x$ $y \in X$

$\exists\, U(x)$ neighb. of x in $\mathcal{T}$ s.t. $y \in \overline{U(x)}$

3) $\forall x \in X$ $\quad \bigcap \{ \mathcal{U} \in \mathcal{T} \mid x \in \mathcal{U} \} = \{x\}$

4) $\Delta \subset X \times X$ diagonal is closed in $X \times X$

1) $\Rightarrow$ 2) $\quad x \neq y \quad \exists\, U(x), V(y) \quad U(x) \cap V(y) = \emptyset$

hence $y \notin \overline{U(x)}$ since $\exists$ open sets around y not intersecting $U(x)$

2) $\Rightarrow$ 3) $\forall y \neq x \quad \exists\, U(x)$ s.t. $y \notin \overline{U(x)}$

so $y \notin \bigcap \{ \mathcal{U} \in \mathcal{T} \mid x \in \mathcal{U} \}$ no $y \neq x$ can be in this inters so contains only x

3) $\Rightarrow$ 4) show Δ^c is open $\quad (x, y) \in \Delta^c$ iff $x \neq y$

$\Rightarrow \exists\, U, V$ open $\quad x \in U \quad y \in V \quad U \cap V \neq \emptyset$

$\Rightarrow U \times V \subseteq \Delta^c$

Properties: • each subspace $Y \subseteq X$ of Hausdorff space is Hausdorff

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- $X = \prod_\alpha X_\alpha$ is Hausdorff iff all X_α are Hausdorff

$x \neq y \in X \Rightarrow x_\alpha \neq y_\alpha$ for some α

$\exists\, U_\alpha, V_\alpha \subseteq X_\alpha \quad x_\alpha \in U_\alpha \quad y_\alpha \in V_\alpha \quad U_\alpha \cap V_\alpha = \phi$

$\Rightarrow U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$ & $V_\alpha \times \prod_{\beta \neq \alpha} X_\beta$

- $f: X \to Y$ open map & injective

if X hausdorff $\Rightarrow$ Y hausdorff

$x \neq y \in X \Rightarrow f(x) \neq f(y) \in Y$ injective

$x \in U \quad y \in V \Rightarrow f(U), f(V)$ open

$U \cap V = \phi \qquad f(U) \cap f(V) = \phi$ (injective)

<u>Compactness</u>

$(X, \mathcal{T})$ (Hausdorff) topological space <u>compact</u>

if $\forall\, \mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ open covering of X $\left(X = \bigcup_{\alpha \in I} U_\alpha\right)$

$\exists$ a finite subcovering $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

- Example X w/ discrete topology $\mathcal{T} = \mathcal{P}(X)$ compact iff X is finite (covering by points)

- in any $(X, \mathcal{T})$ finite subsets are always compact

Equivalent properties:

1) X compact
2) finite intersection property: $\forall \{F_\alpha\}_{\alpha\in I}$ $F_\alpha \in \mathcal{C}$ closed sets

such that $\bigcap_{\alpha\in I} F_\alpha = \phi$

$\exists$ finite subcollection $F_{\alpha_1}, \dots, F_{\alpha_n}$ s.t.

$$\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$$

(dual to compactness by open sets)

Properties: • $f: X \to Y$ continuous surjective

if X compact then Y also compact

in fact: $\{U_\alpha\}_{\alpha\in I}$ open covering of Y then $\{f^{-1}(U_\alpha)\}_\alpha$ is an open covering of X

$\Rightarrow \exists$ $f^{-1}(U_{\alpha_1}) \dots f^{-1}(U_{\alpha_n})$ s.t. $X = f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n})$

$\Rightarrow$ $Y = U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$

• $Y \subseteq X$ with Y compact & X hausdorff (hence Y also hausdorff)

$\Rightarrow$ Y closed in $\mathcal{T}$ of X

in fact: Y^c open: pick $x \in Y^c$ $\forall y \in Y$ $\exists$ $U_y(x)$ $V(y)$ $x \in U_y(x)$ $y \in V(y)$
$U_y(x) \cap V(y) = \phi$

$\{V(y) \cap Y\}_{y\in Y}$ open covering of Y in $\mathcal{T}_Y$ top.

$\Rightarrow$ $\{V(y_1)\cap Y, \dots, V(y_n) \cap Y\}$ finite covering of Y

$\Rightarrow Y \subseteq \bigcup_{i=1}^{n} V(y_i)$ and $\bigcap_{i=1}^{n} U_{y_i}(x) = U(x)$ are disjoint open sets in X

so $U(x) \subseteq Y^c$ open

Reason why convenient to assume Hausdorff hypothesis for compact sets:

- compact subsets of Hausdorff spaces behave "like points" in terms of separation

i.e. Two disjoint ($A \cap B = \emptyset$) A, B compact subsets of X Hausdorff have disjoint neighborhoods

$$A \subseteq U \quad B \subseteq V \quad U \cap V = \emptyset$$

same argument as before

$\forall y \in B \quad \exists \quad \underbrace{U_b(A)}_{\text{as constr. before}} \quad V(b)$ non intersecting open

$\{V(b) \cap B\}_{b \in B} \longrightarrow$ finite subcover $V(b_1) \cap B, \dots V(b_m) \cap V$

$\longrightarrow B \subseteq \bigcup_{i=1}^{n} V(b_i) \qquad \bigcap_{i=1}^{n} U_{b_i}(A)$ open non intersecting

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Useful argument for homeomorphisms:

$$f : X \longrightarrow Y$$

X compact $\quad Y$ Hausdorff

then if f is a continuous bijection it is also a homeomorphism

in fact: need to check f^{-1} continuous i.e.

$\forall U \in \mathcal{T}_X \quad f(U) \in \mathcal{T}_Y$ i.e. f open

equivalently $\forall C \in \mathcal{C}_X$ closed $f(C) \in \mathcal{C}_Y$ closed

1) closed $C \subseteq X$ inside compact is compact
2) $f(C)$ image of compact is compact
3) $f(C)$ compact in Hausdorff is closed

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- closed $C \subseteq X$ inside compact X is compact

take $\{B_\alpha\}_{\alpha \in I}$ closed subsets of C in $\mathcal{T}_X$ w/ $\bigcap_\alpha B_\alpha = \phi$

since $C \subseteq X$ closed the B_α also closed in $\mathcal{T}_X$ ambient top.

X compact $\Rightarrow \exists\, B_{\alpha_1}, \ldots, B_{\alpha_n}$ $\bigcap_{i=1}^n B_{\alpha_i} = \phi$

hence C is also compact

- <u>Tychonoff</u> $X = \prod_{\alpha \in I} X_\alpha$ if compact iff all X_α compact

$(\Rightarrow)$ if X compact since $\pi_\alpha : X \to X_\alpha$ contin. also each $X_\alpha = \pi_\alpha(X)$ compact

just sketch: not precise

$(\Leftarrow)$ open covering $\{U_j\}_j$ of X

can always refine to gen. of top. $U_{\alpha_{j_1}} \times \cdots \times U_{\alpha_{j n_j}} \times \prod X_{\alpha_j}$

U_{α_j} — $U_{\alpha_{jk}}$, X_{α_j} is open covering on X (varying j α fixed)

$\exists$ finite subcovering $U_{\alpha_1} \ldots U_{\alpha_n}$

for all but fin many α these $= X_\alpha$

so $\Rightarrow$ finite subcovering of $\{U_j\}$

- $f : X \to Y$ Y hausdorff & X compact

continuous iff $\Gamma(f) \subset X \times Y$ is closed in $X \times Y$

$X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$

if $\Gamma(f)$ closed then f cont: take $C \subseteq Y$ closed $\Rightarrow \pi_Y^{-1}(C) \cap \Gamma(f)$ closed in prod-top.

$\mathcal{T}_{\Gamma(f)}$ induced by

$\Rightarrow$ in prod top because $\Gamma(f)$ closed

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then $\pi_X(\pi_Y^{-1}(C)\cap\Gamma(f))$ is image in X

need to show π_X is a closed map

(*) comp. $A\times B$ Hausd. $\pi: A\times B\to B$ proj. "parallel to comp. factor" is always closed map

then $f^{-1}(C) = \pi_X(\pi_Y^{-1}(C)\cap\Gamma(f))$ closed in X

so f continuous

other direction: if f contin. map $f: X\to Y$ to a Hausdorff space then $\Gamma(f)$ closed

(**)

show that

(*) $F\subset A\times B$ closed $\Rightarrow$ $B\setminus\pi(F)$ open

take $\overset{\in}{x}$ $(A\times\{x\})\cap F=\emptyset$ so (a,x) has neighb. $U(a)\times V_a(x)$ not intersecting F (A compact, $A\times\{x\}$ comp.)

open cov.

fin subcov. $U(x_i)\cap V_{x_i}(x)$

$\Rightarrow \bigcap V_{x_i}(x)$ neigh. of x not intersecting $\pi(F)$

(**)

contin. $f: X\to Y$ Y Hausdorff

$\Gamma(f)$ = preimage of $\Delta\subset Y\times Y$ under map

$(x,y)\longmapsto(f(x),y)$ continuous

Y Hausdorff $\Leftrightarrow$ $\Delta\subset Y\times Y$ closed $\Rightarrow$ $\Gamma(f)$ closed for $f:X\to Y$ cont.