Topology $\boxed{6}$

Connectedness $(X, \mathcal{T})$ X connected

if $X \neq U \cup V$ with U, V open, $U, V \neq \emptyset$, $U \cap V = \emptyset$

cannot be decomposed as a union of two open nonempty sets that do not intersect

e.g. $X = \{0,1\}$ $\mathcal{T} = \{\emptyset, \{0\}, \{0,1\}\}$ X is connected

$X = \{0\} \cup \{1\}$ only possible decomp into two nonempty non-intersecting sets

$\{0\}$ is open but $\{1\}$ is not open $\Rightarrow X$ connected

but $X = \{0,1\}$ $\mathcal{T} = \mathcal{D} = P(X) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

discrete topology $\Rightarrow X$ not connected

now $X = \{0\} \cup \{1\}$ two nonempty open sets

e.g. $X = \mathbb{R}$ w/ Euclidean topology

only connected sets are intervals (a,b)

$-\infty \leq a, b \leq +\infty$

check: A connected $\Rightarrow$ interval!

if not interval then $\exists a, b \in A$ st. $\exists c$ $a < c < b$ and $c \notin A$ but then $A \cap \{x \mid x < c\} \cup A \cap \{x \mid x > c\}$ is a decomposition $\Rightarrow$ not connected

interval $\Rightarrow$ connected: $Y = A \cup B$ nonempty open $A \cap B = \emptyset$

$\exists a \in A$ $b \in B$

say $a < b$

let $\alpha = \sup\{x \mid [a, x) \subset A\}$

since Y = interval $\Rightarrow \alpha \in Y$

also $\alpha \in \overline{A}_Y$ closure in induced topology $\mathcal{T}_Y$

but in $\mathcal{T}_Y$ if $Y = A \cup B$ decomp. then A, B both open and closed in $\mathcal{T}_Y$ (complement of A is B open and conversely)

since $A = \overline{A}_Y$

$\Rightarrow \alpha \in A$ but since A open

$\Rightarrow (\alpha - \varepsilon, \alpha + \varepsilon) \subset A$ for some $\varepsilon > 0$

contradicts def. of α $\Rightarrow Y$ interval must be connected

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Equivalently
1) Y connected
2) the only subsets $A \subseteq Y$ both open and closed are Y & $\emptyset$
3) no continuous function $f: Y \to \{0, 1\}$ is surjective (w/ discrete top)

$f: X \to Y$ continuous map $(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$

if X connected then $f(X)$ connected

in fact: suppose $f(X)$ not connected $f(X) = U \cup V$ open nonempty non-intersecting

$\Rightarrow f^{-1}(U), f^{-1}(V)$ open nonempty non-intersecting in X

and $X = f^{-1}(U) \cup f^{-1}(V)$ would not be connected

(equivalently if $f(X)$ not conn $\exists\, g: f(X) \to \{0,1\}$ cont. + surj $\Rightarrow g \circ f: X \to \{0,1\}$ cont + surj)

$(X, \mathcal{T})$ $A \subseteq X$ connected

$\Rightarrow$ any set Y s.t. $A \subseteq Y \subseteq \bar{A}$ is also connected (hence $\bar{A}$ connected if A is)

in fact: show that

$f: Y \to \{0,1\}$ cont. cannot be surjective

Know $f|_A : A \to \{0,1\}$ not surjective

$Y = \bar{A} \cap Y = \bar{A}_Y$ closure in induced $\mathcal{T}_Y$

then $f(Y) \subseteq f(\bar{A}_Y) \subseteq f(\bar{A}) \subseteq \underbrace{\overline{f(A)}}_{} = f(A)$

$\Rightarrow$ not surjective

$\{0,1\}$ w/ discr. top. all sets already closed

Connected components

a non-connected space X has a unique decomposition into connected components

$x \in X$ $C(x) \subseteq X$ is largest connected set $\subseteq X$ containing x

connected component of x — i.e. union of all connected subsets of X that contain pt x

Note: $C_\alpha \subseteq X$ connected and $x \in C_\alpha \ \forall \alpha \Rightarrow \bigcup_\alpha C_\alpha = C$ also connected

check: $f: C \to \{0,1\}$ continuous

$f|_{C_\alpha} : C_\alpha \to \{0,1\}$ cont & C_α conn. $\Rightarrow f|_{C_\alpha}$ not surjective

since $x \in C_\alpha \ \forall \alpha$ $f(x) = f|_{C_\alpha}(x)$ for all α $\Rightarrow f|_{C_\alpha}(x)$ all same elt of $\{0,1\}$ $\forall \alpha$

$\Rightarrow f$ not surj.

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Product topology X_α $\qquad \prod_\alpha X_\alpha = X$

$x = (x_\alpha)$ prod. topology $\qquad \pi_\alpha : X \longrightarrow X_\alpha$ projections

sub-basis $\{ \pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{T}_{X_\alpha} \}$

i.e. basis $\bigcap_{i=1}^{N} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = U_{\alpha_1} \times \dots \times U_{\alpha_N} \times \prod_{\alpha \notin \{\alpha_1 \dots \alpha_N\}} X_\alpha$

(e.g. $\prod_\alpha U_\alpha$ w/ $U_\alpha \neq X_\alpha$ for ∞-many coord's not an open set)

smallest topology in which all $\pi_\alpha : X \to X_\alpha$ are continuous

in this topology the π_α are also open

$\prod_\alpha X_\alpha$ w/ product topology is connected iff X_α connected for all α

- $X = \prod_\alpha X_\alpha$ connected $\Rightarrow$ X_α connected because $\pi_\alpha : X \to X_\alpha$ continuous

- X_α connected for all α:

take $x \in X = \prod_\alpha X_\alpha$ and the set

$D_x = \{ y \in X \mid x \,\&\, y$ differ in only finitely many coordinates $\}$

$x = (x_\alpha)$ $\quad y = (y_\alpha)$ $\qquad$ ($x_\alpha = y_\alpha$ for all but fin. many α)

Claim 1: D_x dense in X

given this: take $C(x)$ union of all conn. subsets of X containing x

Claim 2: $D_x \subseteq C(x)$ $\qquad \Rightarrow X = \overline{D_x} \subseteq \overline{C(x)}$ connected $\Rightarrow X = \overline{C(x)}$ conn.

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Claim 1: D_x dense in X ← an equivalent condition to dense: each nonempty basis open set contains an element of D

Claim 2: $D_x \subseteq C(x)$

Show two pts x, y differing in fin. many components are in a same connected set C

basis open sets:

$U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \notin \{\alpha_i\}} X_\alpha$

$\Downarrow$

$y = (x_\alpha)_{\alpha \notin \{\alpha_i\}} \cup (y_{\alpha_1}, \ldots, y_{\alpha_n})$

if $y_{\alpha_1} \in U_{\alpha_1} \ldots y_{\alpha_n} \in U_{\alpha_n}$

if $x_{\alpha_0} \neq y_{\alpha_0}$ for just one α_0

$x, y \in \prod_{\alpha \neq \alpha_0} \{x_\alpha\} \times X_{\alpha_0}$ connected because X_{α_0} conn.

X $\quad x \in X \quad$ $C(x)$ ~~com~~. comp. containing x

$\Rightarrow \{C(x)\}$ partition of X

gives decomposition of X into connected component

(equiv. class $x \sim y$ if $C(x) = C(y)$)

$f: X \to Y$ continuous $\Rightarrow$ connected comp. of X contained in conn. comp of Y

$C \subset X$ conn. comp $\quad f(C) \subseteq C'$ some C' conn comp of Y

$(X, \mathcal{T})$ <u>locally</u> connected if $\exists \mathcal{B}$ basis of topology $\mathcal{T}$ s.t. all $B \in \mathcal{B}$ are connected

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X locally connected if $\forall \mathcal{U} \in \mathcal{T}$ the connected components of $\mathcal{U}$ are open sets

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- $C \subseteq \mathcal{U}$ conn. comp $\quad B \in \mathcal{B}$ basis of conn. open sets

  $\forall x \in C \quad \exists B \quad x \in B \subseteq \mathcal{U}$ and B connected so $B \subseteq C$

- conversely: the set (assume $C \subseteq \mathcal{U}$ conn. comp $\Rightarrow$ open) $\Rightarrow C$ is open

  $\mathcal{B} = \{$all connected comp of all open sets $\mathcal{U}\}$
  is a basis for the topology $\Rightarrow X$ loc. conn.

$X = \prod_\alpha X_\alpha$ loc. conn. iff all X_α loc. conn. & all but finitely many conn

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A path in $(X, \mathcal{T})$

$f: [0,1] \longrightarrow X$ continuous
(Euclidean)

a path connecting $x, y \in Y \subseteq X$

$f: [0,1] \to X$ cont. s.t. $f(0) = x \quad f(1) = y$

$f(t) \in Y \quad \forall t$

$Y \subseteq X$ path connected if $\forall x, y \; \exists$ path connecting them in Y

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path connected $\Rightarrow$ connected $\qquad f(I)$ connected since I is & f contin.

$\not\Leftarrow$

if $X = U \cup V \quad U, V \neq \emptyset \quad U \cap V \neq \emptyset$

take $t_+ = \sup\{t \mid f(t) \in U\} \subseteq C(x) = U \qquad$ then $\exists x \in U \quad y \in V$

$t_- = \inf\{t \mid f(t) \in V\} \subseteq C(y) = V \qquad$ suppose $f(0) = x \quad f(1) = y$

$f(I) \subseteq C(x) \Rightarrow f(I) \subseteq C(y)$ ... not ... $f(t) \in X$ ...