Topology $\boxed{1}$

"set of all sets that do not contain themselves as elements" contains itself?

Set theory basics

$a \in A$ element in set

sets $A = B$ iff $A \subseteq B$ and $B \subseteq A$

$A \subseteq B$: ($\forall a$ $a \in A \Rightarrow a \in B$)

$\subseteq$ and $\subset$ but
inclusion proper

Note: often some use $\subset$ for $\subseteq$ and $\subsetneq$ for proper

$\emptyset$ empty set (no elements)
$\{x\}$ set w/ one element x

Boolean algebra of sets: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- disjoint: $A \cap B = \emptyset$
- $A \cup \emptyset = A$ $A \cap \emptyset = \emptyset$
- always $A \cap B \subseteq A \subseteq A \cup B$

  $A \subseteq C$ and $B \subseteq D \Rightarrow$ $A \cup B \subseteq C \cup D$
  $A \cap B \subseteq C \cap D$

- formal operations / properties

  idempotent: $A \cup A = A = A \cap A$ for all sets A

  associative: $A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$

  commutative: $A \cup B = B \cup A$ $A \cap B = B \cap A$

  distributive: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

check this: $x \in A \cap (B \cup C)$ means $x \in A$ and ($x \in B$ or $x \in C$)

so ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$x \in A \cap B$ $x \in A \cap C$

logic		sets
$\wedge$ and	$\longleftrightarrow$	$\cap$
$\vee$ or	$\longleftrightarrow$	$\cup$

- difference $A \setminus B = \{x \in A \mid x \notin B\}$
- complement $A \subseteq C$ complement of A in C $= C \setminus A$

- some <u>ambient larger</u> set E fixed

  subsets of E complement $A^c = \{x \in E \mid x \notin A\}$
  $= E \setminus A$

$A \cap A^c = \emptyset$ $\quad$ $A \cup A^c = E$

$(A^c)^c = A$ $\quad$ $\emptyset^c = E$ $\quad$ $E^c = \emptyset$

$A \subseteq B$ iff $B^c \subseteq A^c$

- <u>de Morgan laws</u> $\quad (A \cup B)^c = A^c \cap B^c$

  $(A \cap B)^c = A^c \cup B^c$

  logic $\quad$ set theory
  $\neg$ $\longleftrightarrow$ c
  negation $\quad$ complement

first $x \in (A \cup B)^c$ means $\{x \in E \mid x \notin A \cup B\} = x \notin \{y \in A \text{ or } y \in B\}$
$\Leftrightarrow x \notin A$ and also $x \notin B$

second same way

$\{x \notin A \ \wedge \ x \notin B\} \Leftrightarrow A^c \cap B^c$

- a <u>Boolean algebra</u> $\mathcal{B}$
  is a set w/ two ~~operations~~ binary (two inputs one output)
  $\vee, \wedge$ and a unary op (one input one output) $\neg$

  s.t. $\exists\, 0, 1$ elements $\quad a \vee 0 = a \quad a \wedge 1 = a$

  $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

  $a \wedge a' = 0 \quad a \vee a' = 1$

  $0 = \emptyset$
  $1 = E$
  $\wedge = \cap$
  $\vee = \cup$
  $\neg = c$

  $\vee, \wedge$
  $+, \cdot$ w/ different properties from addition & mult. of numbers

(note associativity follows not part of axioms)

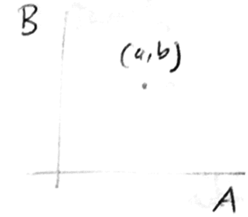
duality: symmetry $+, \cdot, 0, 1 \longleftrightarrow \cdot, +, 1, 0$

• Cartesian products

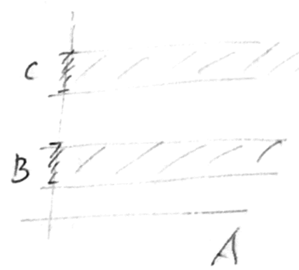
ordered pairs (a,b) $\quad a \in A \quad b \in B$

property: $(a,b) = (c,d)$ iff $a = c$ and $b = d$ $\quad \rightarrow$ hence $(a,b) = (b,a)$ iff $a = b$

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$



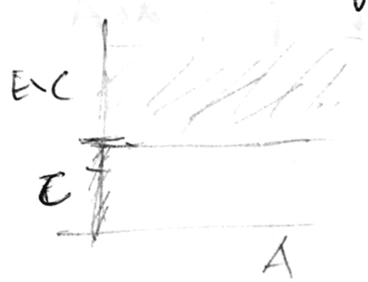
- $A \times B = \emptyset$ iff $A = \emptyset$ and $B = \emptyset$
- if nonempty $\quad C \times D \subseteq A \times B$ iff $C \subseteq A$ and $D \subseteq B$
- $A \times B = B \times A$ iff $A = B$ (again if nonempty)

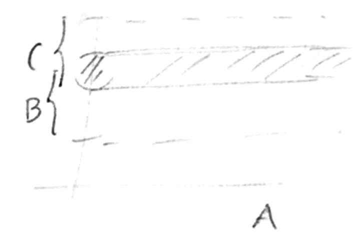


- distributivity: $A \times (B \cup C) = A \times B \cup A \times C$

$A \times (B \cap C) = A \times B \cap A \times C$

$A \times (E \setminus C) = A \times E \setminus A \times C$





• Families of sets

I a set of indices

$\alpha \in I \qquad A_\alpha$ a set for each index $\alpha \in I$

$\{A_\alpha\}_{\alpha \in I}$ a collection/family of sets

- can assume all A_α are subset of a same ambient set E (can always take $E = \bigcup_\alpha A_\alpha$)

- union $\quad \bigcup_{\alpha \in I} A_\alpha = \{ x \mid x \in A_\alpha \text{ for some } \alpha \in I \}$

$\exists \alpha \in I \text{ s.t. } x \in A_\alpha$

- intersection $\quad \bigcap_{\alpha \in I} A_\alpha = \{ x \mid x \in A_\alpha \text{ for all } \alpha \in I \}$

$= \{ x \mid \forall \alpha \in I : x \in A_\alpha \}$

$\wedge$

$\vee$

- $\left(\bigcup_\alpha A_\alpha \right)^c = \bigcap_\alpha A_\alpha^c \qquad \left(\bigcap_\alpha A_\alpha \right)^c = \bigcup_\alpha A_\alpha^c$

④

- "power set" or "set of parts" or "set of subsets"

A set $\quad \mathcal{P}(A) = \{ B \mid B \subseteq A \}$ $\quad$ set of all possible subsets of A

also denoted 2^A because if $\quad 2^A$ or $\{0,1\}^A = \{ \cancel{f} : A \to \{0,1\} \}$

- A finite set of n elements $A = \{a_1, \dots, a_n\}$

then $\mathcal{P}(A)$ finite set of 2^n elements

to specify a subset $B \subseteq A$: go through list $a_1, \dots, a_n$ of elements of A and for each answer yes/no to "is it in B?"

so all possible subsets are all possible way of assigning a yes/no (0/1) value to each point $a \in A$

there are 2^n ways of doing that

- properties: $\bigcap_\alpha \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_\alpha A_\alpha)$ $\quad$ check this (exercise)

<u>but</u> $\quad \bigcup_\alpha \mathcal{P}(A_\alpha) \subseteq \mathcal{P}(\bigcup_\alpha A_\alpha)$

take $A_1 = \{1\}$ $\quad A_2 = \{2\}$ $\quad \mathcal{P}(A_1) = \{\emptyset, A_1\}$

$\mathcal{P}(A_2) = \{\emptyset, A_2\}$

$\mathcal{P}(A_1) \cup \mathcal{P}(A_2) = \{\emptyset, A_1, A_2\}$

$A_1 \cup A_2 = \{1, 2\}$ $\quad \mathcal{P}(A_1, A_2) = \{\emptyset, A_1, A_2, A_1 \cup A_2\}$

- functions and "correspondences"

$f : X \to Y$ $\quad$ sets X, Y

subset of the prod. $\Gamma(f) \subset X \times Y$ (graph of f) $\quad$ s.t. $\forall x \in X$ $\quad \{x\} \times Y \cap \Gamma(f)$ either empty or single pt $(x, y = f(x))$

- set $\{x\in X \mid \{x\}\times Y \cap \Gamma(f) \neq \phi\} = \mathrm{Dom}(f)$ domain of f

  $\{y\in Y \mid X\times\{y\} \cap \Gamma(f) \neq \phi\} = \mathrm{Range}(f)$ range

Sometimes write $f: X\to Y$ assuming $\mathrm{Dom}(f) = X$

- subsets $\Gamma \subset X\times Y$ where $\{x\}\times Y \cap \Gamma$ can contain more than one pt.
  are <u>not</u> functions but more general <u>correspondences</u> (multivalued functions)

- if $\mathrm{Range}(f) = \{y\}$ a single pt in Y constant map
- if $\mathrm{Dom}(f) = X$ and $X\times\{y\}\cap\Gamma(f) \neq \phi$ implies $X\times\{y\}\cap\Gamma(f) \exists$ single pt. x: $= \{(x, f(x))\}$

  f is injective
- if $\mathrm{Range}(f) = Y$ f surjective
- injective & surjective = <u>bijective</u> ← isomorphism on sets
- induced map on subsets : <u>preimage</u> $B\subset Y$

  $f^{-1}(B) = \{x\in X \mid f(x)\in B\}$

  (note f does not have to be invertible)

  $f: X\to Y$

  e.g. $f =$ constant $= y$

  $f^{-1}(B) = \phi$ if $y\notin B$

  $f^{-1}(B) = X$ if $y\in B$

  $f^{-1}: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$

  $B \longmapsto f^{-1}(B)$

  taking preimages of sets works better than taking images (preserves union not inters.)

  $f(\bigcap_\alpha A_\alpha) \subseteq \bigcap_\alpha f(A_\alpha)$

  $A = [0,1]$, $B = [2,3]$, $f(x) = 1$ constant

  $f(A\cap B) = \phi$ but $f(A)\cap f(B) = \{1\}$

properties:

$$f^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha f^{-1}(B_\alpha)$$

$$f^{-1}\left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha f^{-1}(B_\alpha)$$

$\left.\right\}$ $f(x)\in\bigcap_\alpha B_\alpha$

$\iff f(x)$ in each B_α

$\iff x$ in each $f^{-1}(B_\alpha)$

$$f^{-1}(E\setminus B) = f^{-1}(E)\setminus f^{-1}(B)$$

(6)

- **covering**: X set $\{A_\alpha\}_{\alpha\in I}$ family of subsets

$A_\alpha \subseteq X$

is a covering of X if $\bigcup_\alpha A_\alpha = X$

- **patching functions**: $f_\alpha : A_\alpha \to Y$ s.t. $\forall \alpha, \beta \in I$

$$f_\alpha\big|_{A_\alpha\cap A_\beta} = f_\beta\big|_{A_\alpha\cap A_\beta} \quad \Bigg\{ \quad \underbrace{f_\alpha\big|_{A_\alpha\cap A_\beta}}_{\text{restriction}} : A_\alpha\cap A_\beta \to Y \qquad f_\beta\big|_{A_\alpha\cap A_\beta} : A_\alpha\cap A_\beta \to Y$$

$$\Rightarrow \exists\, f : X \to Y \quad \text{s.t.} \quad f\big|_{A_\alpha} = f_\alpha$$

just because setting

$f(x) = f_\alpha(x)$ for $x\in A_\alpha$ is well defined

(if $x\in A_\beta$ same value $f(x)$)

- **partition**: X set $\{A_\alpha\}_{\alpha\in I}$ covering of X

is a partition of X iff $A_\alpha \cap A_\beta = \emptyset$
for all $\alpha\neq\beta\in I$

- **Relations**: Equivalence relations set X

$R \subset X\times X$ subset of Cartesian product
(= binary relation on X)

w/ properties:

R equivalence relation
xRy or $x\overset{R}{\sim}y$

- $\forall x\in X$ $(x,x)\in R$ reflexive
- $\forall x,y\in X$ $(x,y)\in R \Leftrightarrow (y,x)\in R$ symmetric
- $\forall x,y,z\in X$ if $(x,y)\in R$ and $(y,z)\in R \Rightarrow (x,z)\in R$

- equivalence classes $R \subset X \times X$ equivalence relation

$x \in X$ $\quad Rx = \{ y \in X \mid x \overset{R}{\sim} y \}$ equiv class of x

i.e. $(x,y) \in R$ — also write $[x]_R$

properties:

- $\bigcup_{x \in X} Rx = X$ (note: each $x \in Rx$ because reflexive so $\bigcup_x Rx \supseteq \bigcup_x \{x\} = X$)

- if $(x,y) \in R \Rightarrow Rx = Ry$ (transitivity: if $u \in Rx$, $u \overset{R}{\sim} x$, $y \overset{R}{\sim} x \Rightarrow u \overset{R}{\sim} y$, $x \in Ry$; same other direction)

- if $(x,y) \notin R \Rightarrow Rx \cap Ry = \emptyset$ (if $\xi \in Rx \cap Ry \neq \emptyset$ then $\xi \underset{R}{\sim} x$ and $\xi \underset{R}{\sim} y \underset{\text{trans}}{\Rightarrow} x \underset{R}{\sim} y$ contradiction)

these properties imply that

- the set of equivalence classes $\{Rx\}_{x \in X}$ is a partition of X

- an element $u \in Rx$ a "representative" of the class $Rx = [x]_R$

- Set of equivalence classes $\{Rx\}_{x \in X}$ is called quotient set of X by the equivalence relation R $\quad X/R$

map $\pi_R : X \longrightarrow X/R$ — projection map

$x \longmapsto Rx$ — surjective (not injective) more than one el. in each class

8 $f: X \to Y$ function

equivalence relations $R \subset X \times X$ and $R' \subset Y \times Y$

f is <u>compatible</u> w/ equivalence relations if

$$(x,y) \in R \implies (f(x), f(y)) \in R'$$

then $\exists!$ map $h: X/R \to Y/R'$ of quotient spaces

s.t.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi_R"'] & Y \arrow[d, "\pi_{R'}"] \\
X/R \arrow[r, "h"] & Y/R'
\end{tikzcd}$$

diagram commutes:

$$h \circ \pi_R = \pi_{R'} \circ f$$

map <u>induced</u> on the quotients by f

(f passes to the quotient)

to <u>see this</u>:

given $Rx \in X/R$ want to define $h(Rx)$

choose a representative $u \in Rx$

and define $h(Rx) = \pi_{R'}(f(u))$ for this to be well defined need to know <u>independent</u> of which representative $u \in Rx$ chosen

$v \in Rx$ $v \neq u$ another one: $u, v \in Rx \implies u \overset{R}{\sim} v$

(because $u \underset{R}{\sim} x$ and $v \underset{R}{\sim} x$)

f compatible w/ relations

$$\implies (u \overset{R}{\sim} v) \implies (f(u) \overset{R'}{\sim} f(v))$$

$$\implies \pi_{R'}(f(u)) = R'f(u) = R'f(v) = \pi_{R'}(f(v))$$

So value $h(Rx)$ is well defined, h is a function