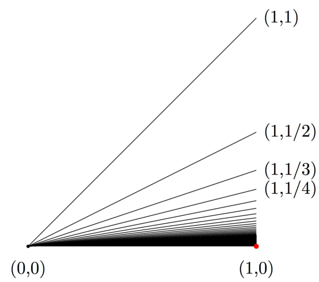
MIDTERM

MA109A: FALL 2021

(1) Consider the set $X \subset \mathbb{R}^2$ consisting of the union

$$X = \{(1,0)\} \cup \bigcup_{n \geq 1} L_n$$

with L_n the segment from $(0,0)$ to $(1, \frac{1}{n})$, as in the figure. Show that X is connected but not path connected.



(2) In the unit square $\mathcal{I} \times \mathcal{I}$, with $\mathcal{I} = [0,1]$, construct two connected disjoint sets X and Y such that X contains the points $(0,0)$ and $(1,1)$ and Y contains the points $(0,1)$ and $(1,0)$. Can these sets be path connected?

(3) The middle-third Cantor set C is the subset of $[0,1]$ obtained by successively removing the middle third of each interval, that is, $C = \cap_{n \geq 0} C_n$ with $C_0 = [0,1]$ and $C_n = \frac{1}{3} C_{n-1} \cup (\frac{2}{3} + \frac{1}{3} C_{n-1})$ or equivalently

$$C = [0,1] \smallsetminus \bigcup_{n \geq 0} \bigcup_{k=0}^{3^n - 1} \left(\frac{3^k + 1}{3^{n+1}}, \frac{3^k + 2}{3^{n+1}} \right).$$

- Represent points $x \in C$ as $x = \sum_{n=1}^{\infty} a_n 3^{-n}$ with $a_n \in \{0, 2\}$.
- Show that the maps $f : C \to [0,1]$ and $g : C \to C \times C$ given by

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} a_n 2^{-n}, \quad g(x) = (g_1(x), g_2(x)) = (\sum_{n=1}^{\infty} a_{2n-1} 3^{-n}, \sum_{n=1}^{\infty} a_{2n} 3^{-n})$$

are continuous surjections.

Date: due: by end of day Tuesday November 2.

- Show that the map $F : C \to [0,1]^2$ given by $F(x) = (f \circ g_1(x), f \circ g_2(x))$ is also a continuous surjection and that it is the restriction to C of a continuous surjection $\tilde{F} : [0,1] \to [0,1]^2$ (a space-filling curve).
- Construct a bijective map from C to the interval $[0,1]$.
- Consider both C and $[0,1]$ inside $\mathbb{R}$ with the Euclidean topology, endowed with the induced topology: can the bijection be constructed so that it is (a) continuous? (b) open? (c) a homeomorphism?
- Take the set $\{0,1\}$ with the discrete topology and form the infinite product $X = \prod_{n\in\mathbb{N}}\{0,1\} = \{0,1\}^{\mathbb{N}}$, with the product topology $\mathcal{T}$. Show that
$$d(x,y) = \sup_{n\in\mathbb{N}} 2^{-n}|x_n - y_n|$$
is a metric on $(X, \mathcal{T})$.
- Show that C is homeomorphic to X.

(4) let $(X, \mathcal{T})$ be a topological space. Given any subset $A \subseteq X$ let $\mathcal{Q}_A$ be the collection of sets obtained by repeatedly applying to A the operations of closure and complement in any order,
$$\mathcal{Q}_A = \{A, A^c, \overline{A}, \overline{A}^c, \overline{A^c}, \overline{\overline{A}^c}, \overline{A^c}^c, \cdots\}$$
- Show that, for any $A \subseteq X$, there are at most 14 distinct sets in the collection $\mathcal{Q}_A$.
- For $X = \mathbb{R}$ with the Euclidean topology, find a set $A \subset \mathbb{R}$ for which $\mathcal{Q}_A$ contains exactly 14 different sets.
- Let $X = \{1, \ldots, 7\}$ with the topology with basis
$$\mathcal{B} = \{\emptyset, X, \{1\}, \{6\}, \{1,2\}, \{3,4\}, \{5,6\}\}.$$
check that the subset $A = \{1,3,5\}$ of X also produces 14 different sets in $\mathcal{Q}_A$.

(5) Let **CHaus** be the category of compact Hausdorff topological spaces, where the objects are topological spaces $(X, \mathcal{T}_X)$ that are both compact and Hausdorff and morphisms are continuous maps between them.
- Show that if $(X, \mathcal{T}_X)$ is compact and Hausdorff then in any topology $\mathcal{T}'_X$ coarser than $\mathcal{T}_X$ the space X would no longer be Hausdorff and in any topology $\mathcal{T}''_X$ finer than $\mathcal{T}_X$ it would no longer be compact.
- Show that, unlike **Top**, the category **CHaus** is a balanced category.
- Show that the forgetful functor F : **CHaus** $\to$ **Sets** reflects isomorphisms (that is, if $F(f)$ is an isomorphism in **Sets** then f is an isomorphism in **CHaus**).

(6) On $\mathbb{N} \times \mathbb{N}$ consider the function

$$d(n,m) = \begin{cases} 1 + \frac{1}{n+m} & m \neq n \\ 0 & n = m. \end{cases}$$

- Show that $d(n,m)$ is a metric on $\mathbb{N}$ and that $(\mathbb{N}, d)$ is complete.
- Show that $(\mathbb{N}, d)$ is not compact by constructing a family of closed sets that does not satisfy the finite intersection condition.