Homotopy Theoretic and Categorical Models of Neural Information Networks

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• based on ongoing joint work with Yuri I. Manin (Max Planck Institute for Mathematics)

- Yuri Manin, Matilde Marcolli *Homotopy Theoretic and Categorical Models of Neural Information Networks*, arXiv:2006.15136

• related work:

- M. Marcolli, *Gamma Spaces and Information*, Journal of Geometry and Physics, 140 (2019), 26–55.

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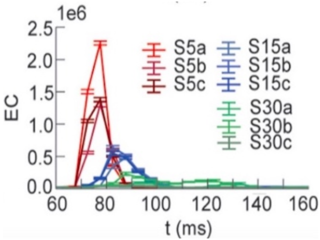
Motivation N.1: Nontrivial Homology

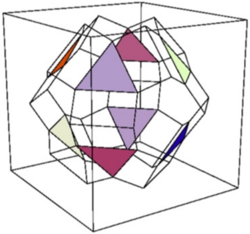
- Kathryn Hess' applied topology group at EPFL: topological analysis of neocortical microcircuitry (Blue Brain Project)



- formation of large number of high dimensional cliques of neurons (complete graphs on N vertices with a directed structure) accompanying response to stimuli
- formation of these structures is responsible for an increasing amount of nontrivial Betti numbers and Euler characteristics, which reaches a peak of topological complexity and then fades
- proposed functional interpretation: this peak of non-trivial homology is necessary for the processing of stimuli in the brain cortex... but why?

Motivation N.2: Computational Role of Nontrivial Homology

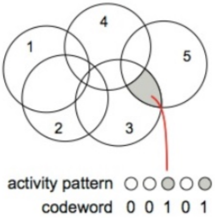
- mathematical theory of concurrent and distributed computing (Fajstrup, Gaucher, Goubault, Herlihy, Rajsbaum, ...)
- initial, final states of processes vertices, $d + 1$ mutually compatible initial/final process states d-simplex

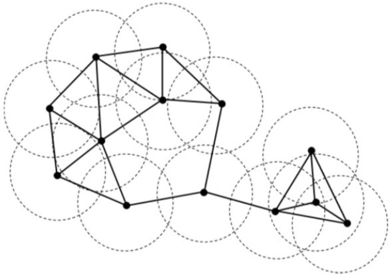


- distributed algorithms: simplicial sets and simplicial maps
- certain distributed algorithms require "enough non-trivial homology" to successfully complete their tasks (Herlihy–Rajsbaum)
- this suggests: functional role of non-trivial homology to carry out some concurrent/distributed computation

Motivation N.3: Neural Codes and Homotopy Types

- Carina Curto and collaborators: geometry of stimulus space can be reconstructed *up to homotopy* from binary structure of the neural code





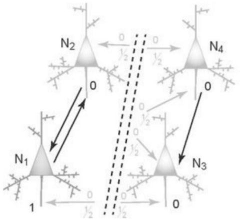
- overlaps between place fields of neurons and the associated simplicial complex of the open covering has the same homotopy type as the stimulus space
- this suggests: the neural code *represents* the stimulus space through homotopy types, hence homotopy theory is a natural mathematical setting

Motivation N.4: Informational and Metabolic Constraints

- neural codes: rate codes (firing rate of a neuron), spike timing codes (timing of spikes), neural coding capacity for given firing rate, output entropy
- metabolic efficiency of a transmission channel ratio $\epsilon = I(X,Y)/E$ of the mutual information of output and input X and energy cost E per unit of time
- optimization of information transmission in terms of connection weights maximizing mutual information $I(X,Y)$
- requirement for homotopy theoretic modelling: need to incorporate constraints on resources and information (mathematical theory of resources: Tobias Fritz and collaborators, categorical setting for a theory of resources and constraints)

Motivation N.5: Informational Complexity

- measures of informational complexity of a neural system have been proposed, such as integrated information: over all splittings $X = A \cup B$ of a system and compute minimal mutual information across the two subsystems, over all such splittings



- controversial proposal (Tononi) of integrated information as measure of consciousness (but simple mathematical systems from error correcting codes with very high integrated information!)
- some better mathematical description of organization of neural system over subsystems from which integrated information follows?

Main Idea for a homotopy theoretic modeling of neural information networks

- Want a space (topological) that describes all consistent ways of assigning to a population of neurons with a network of synaptic connections a concurrent/distributed computational architecture ("consistent" means with respect to all possible subsystems)
- Want this space to also keep track of constraints on resources and information and conversion of resources and transmission of information (and information loss) across all subsystems
- Want this description to also keep track of homotopy types (have homotopy invariants, associated homotopy groups): topological robustness
- Why use category theory as mathematical language? because especially suitable to express "consistency over subsystems" and "constraints over resources"
- also categorical language is a main tool in homotopy theory (mathematical theory of concurrent/distributed computing already knows this!)

Categories of Resources

- mathematical theory of resources
  - B. Coecke, T. Fritz, R.W. Spekkens, *A mathematical theory of resources*, Information and Computation 250 (2016), 59–86. [arXiv:1409.5531]
- Resources modelled by a symmetric monoidal category $(\mathcal{R}, \circ, \otimes, \mathbb{I})$
- objects $A \in \mathrm{Obj}(\mathcal{R})$ represent resources, product $A \otimes B$ represents combination of resources, unit object $\mathbb{I}$ empty resource
- morphisms $f : A \to B$ in $\mathrm{Mor}_{\mathcal{R}}(A, B)$ represent possible conversions of resource A into resource B
- convertibility of resources when $\mathrm{Mor}_{\mathcal{R}}(A, B) \neq \emptyset$

Measuring semigroups of categories of resources (Coecke, Fritz, Spekkens)

- preordered abelian semigroup $(R, +, \succeq, 0)$ on set R of isomorphism classes of $\mathrm{Obj}(\mathcal{R})$ with $A + B$ the class of $A \otimes B$ with unit 0 given by the unit object $\mathbb{I}$ and with $A \succeq B$ iff $\mathrm{Mor}_{\mathcal{R}}(A, B) \neq \emptyset$
- (same for category $\mathcal{C}$ with sum and zero object)
- maximal conversion rate $\rho_{A \to B}$ of resources

$$\rho_{A \to B} := \sup\{ \frac{m}{n} \,|\, n \cdot A \succeq m \cdot B, \;\; m, n \in \mathbb{N} \}$$

  number of copies of resource A are needed on average to produce B
- measuring semigroup: abelian semigroup with partial ordering and semigroup homomorphism $M : (R, +) \to (S, *)$ with $M(A) \geq M(B)$ in S when $A \succeq B$ in R
- satisfy $\rho_{A \to B} \cdot M(B) \leq M(A)$

Summing functors

- $\mathcal{C}$ a category with sum and zero-object (binary codes, transition systems, resources, etc)
- (X, x_0) a pointed finite set and $\mathcal{P}(X)$ a category with objects the pointed subsets $A \subseteq X$ and morphisms the inclusions $j : A \subseteq A'$
- a functor $\Phi_X : \mathcal{P}(X) \to \mathcal{C}$ summing functor if

$$\Phi_X(A \cup A') = \Phi_X(A) \oplus \Phi_X(A') \quad \text{when} \quad A \cap A' = \{x_0\}$$

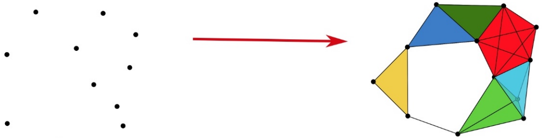
  and $\Phi_X(\{x_0\})$ is zero-object of $\mathcal{C}$
- $\Sigma_{\mathcal{C}}(X)$ category of summing functors $\Phi_X : \mathcal{P}(X) \to \mathcal{C}$, morphisms are *invertible* natural transformations

• Key idea: a summing functor is a *consistent assignment* of *resources* of type $\mathcal{C}$ to *all subsystems* of X so that a combination of independent subsystems corresponds to combined resources

• $\Sigma_{\mathcal{C}}(X)$ parameterizes all possible such assignments

Segal's Gamma Spaces

- construction introduced in homotopy theory in the '70s: a general construction of (connective) *spectra* (generalized homology theories)
- a Gamma space is a functor $\Gamma : \mathcal{F} \to \Delta$ from finite (pointed) sets to (pointed) simplicial sets



- a category $\mathcal{C}$ with sum and zero-object determines a Gamma space $\Gamma_{\mathcal{C}} : \mathcal{F} \to \Delta$
  - for a finite set X take category of summing functors $\Sigma_{\mathcal{C}}(X)$ and simplicial set given by nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of this category

Meaning in homotopy theory (loop-deloop)

- take case where $\mathcal{C}$ is an abelian category, then (Quillen) the higher K-theory $K(\mathcal{C})$ is the K-theory of an infinite loop space
- the category of summing functors $\Sigma_{\mathcal{C}}(X)$ provides a delooping of this infinite loop space (Carlsson)
- a Gamma space defines an associated spectrum, by extending the functor $\Gamma : \mathcal{F} \to \Delta$ to an endofunctor $\Gamma : \Delta \to \Delta$ and applying it to spheres
- when $\mathcal{C} = \mathcal{F}$ with $\Gamma_{\mathcal{F}} : \mathcal{F} \to \Delta$ get the sphere spectrum
- all connective spectra are obtained through this construction for $\mathcal{C}$ a symmetric monoidal category (Thomason)
- hence nerves $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ are topologically very nontrivial

Meaning in our setting

- nerve $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ of category of summing functors organizes all assignments of $\mathcal{C}$-resources to X-subsystems and their transformations into a single topological structure that keeps track of equivalence relations between them (invertible natural transformations as morphisms of $\Sigma_{\mathcal{C}}(X)$ and their compositions become simplexes of the nerve)
- view $\mathcal{N}(\Sigma_{\mathcal{C}}(X))$ as a topological parameterizing space for all such consistent assignments of resources of type $\mathcal{C}$ to subsets of X

From finite sets to networks: directed graphs

- category 2 has two objects V, E and two morphisms $s, t \in \mathrm{Mor}(E, V)$
- $\mathcal{F}$ category of finite sets: objects finite sets, morphisms functions between finite sets
- a directed graph is a functor $G : 2 \to \mathcal{F}$
  - $G(E)$ is the set of edges of the directed graph
  - $G(V)$ is the set of vertices of the directed graph
  - $G(s) : G(E) \to G(V)$ and $G(t) : G(E) \to G(V)$ are the usual source and target maps of the directed graph
- category of directed graphs $\mathrm{Func}(2, \mathcal{F})$ objects are functors and morphisms are natural transformations

Systems organized according to networks

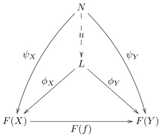
- instead of finite set X want a directed graph (network) and its subsystems
- directed graph as functor $G : 2 \to \mathcal{F}$ and functorial assignment $X \mapsto \Sigma_{\mathcal{C}}(X)$
- $\Sigma_{\mathcal{C}}(E_G)$ summing functors $\Phi_E : \mathcal{P}(E_G) \to \mathcal{C}$ for sets of edges and $\Sigma_{\mathcal{C}}(V_G)$ summing functors $\Phi_V : \mathcal{P}(V_G) \to \mathcal{C}$ for sets of vertices
- source and target maps $s, t : E_G \to V_G$ transform summing functors $\Phi_E \in \Sigma_{\mathcal{C}}(E_G)$ to summing functors in $\Sigma_{\mathcal{C}}(V_G)$

$$\Phi^s_{V_G}(A) := \Phi_{E_G}(s^{-1}(A)) \quad \Phi^t_{V_G}(A) := \Phi_{E_G}(t^{-1}(A))$$

  assigns to a set of vertices $\mathcal{C}$-resources of in/out edges
- categorical statement: source and target maps $s, t : E_G \to V_G$ determine functors between categories $\Sigma_{\mathcal{C}}(E_G)$ and $\Sigma_{\mathcal{C}}(V_G)$ of summing functors, hence map between their nerves

Expressing constraints and optimization in categorical form

- limits and colimits in categories
  - diagram $F : \mathcal{J} \to \mathcal{C}$ and cone N, limit is "optimal cone" (dual version for colimits)



  - special cases of limits and colimits: equalizers, coequalizers
- Example: thin categories $(S, \leq)$ set of objects S and one morphism $s \to s'$ when $s \leq s'$
  - diagram in $(S, \leq)$ is selection of a subet $A \subset S$
  - limits and colimits greatest lower bounds and least upper bounds for subsets $A \subseteq S$
- Key idea: functors compatible with limits and colimits describe constrained optimization

Conservation laws at vertices

- source and target functors $s, t : \Sigma_{\mathcal{C}}(E_G) \rightrightarrows \Sigma_{\mathcal{C}}(V_G)$
- equalizer category $\Sigma_{\mathcal{C}}(G)$ with functor $\iota : \Sigma_{\mathcal{C}}(G) \to \Sigma_{\mathcal{C}}(E_G)$ such that $s \circ \iota = t \circ \iota$ with universal property

$$\begin{array}{ccccc} \Sigma_{\mathcal{C}}(G) & \xrightarrow{\iota} & \Sigma_{\mathcal{C}}(E_G) & \overset{s}{\underset{t}{\rightrightarrows}} & \Sigma_{\mathcal{C}}(V_G) \\ \uparrow \exists u & \nearrow q & & & \\ \mathcal{A} & & & & \end{array}$$

- this is category of summing functors $\Phi_E : P(E_G) \to \mathcal{C}$ with conservation law at vertives: for all $A \in P(V_G)$

$$\Phi_E(s^{-1}(A)) = \Phi_E(t^{-1}(A))$$

  in particular for all $v \in V_G$ have inflow of $\mathcal{C}$-resources equal outflow

$$\oplus_{e:s(e)=v} \Phi_E(e) = \oplus_{e:t(e)=v} \Phi_E(e)$$

- another kind of conservation law expressed by coequalizer

Gamma spaces for networks

- $\mathcal{E}_{\mathcal{C}} : \mathrm{Func}(2, \mathcal{F}) \to \Delta$ with $\mathcal{E}_{\mathcal{C}}(G) = \mathcal{N}(\Sigma_{\mathcal{C}}(G))$ nerve of equalizer of $s, t : \Sigma_{\mathcal{C}}(E_G) \rightrightarrows \Sigma_{\mathcal{C}}(V_G)$ (equalizer of nerves)
- more general types of Gamma networks besides equalizers $\Sigma_{\mathcal{C}}^{eq}(G)$ (and coequalizers)
  - for $G \in \mathrm{Func}(2, \mathcal{F})$ take category $P(G)$ with objects (pointed) subgraphs G'_* of G_* and morphisms (pointed) inclusions $\iota : G'_* \hookrightarrow G''_*$
  - category $\Sigma_{\mathcal{C}}(G)$ of summing functors $\Phi_G : P(G) \to \mathcal{C}$
  - now value of functor $\Phi_G \in \Sigma_{\mathcal{C}}(G)$ on a subnetwork $G' \subset G$ not just sum of values on edges in the subnetwork
  - possible more complicated dependence on network structure (beyond conservation at vertices): general inclusion-exclusion type properties
- focus on equalizer case for simplicity

Category of binary codes

- C be a $[n,k,d]_2$ binary code of length n with $\#C = q^k$
- category $\mathrm{Codes}_{n,*}$ of pointed codes of length n
  - objects codes that contain 0-word $c_0 = (0,0,\ldots,0)$
  - exclude code consisting only of constant words $c_0 = (0,0,\ldots,0)$ and $c_1 = (1,1,1,\ldots,1)$ (for reasons of non-trivial information)
  - morphisms $f : C \to C'$ functions mapping the 0-word to itself (don't require maps of ambient $\mathbb{F}_2^n$)
  - sum as for pointed sets $C \vee C'$ (glued along the zero-word)
  - zero-object: code consisting only of the zero word
  - role of zero-word is like reference point (for neural code, baseline when no activity detected)

• Note: in coding theory often other form of categorical sum (decomposable codes), but changes code length n

$$C \oplus C' := \{(c,c') \in \mathbb{F}_2^{n+n'} \mid c \in C,\ c' \in C'\}$$

neural codes

- $T > 0$ time interval of observation, subdivided into some basic units of time, Δt
- code length $n = T/\Delta t$: number of basic time intervals considered
- number of nontrivial code words: neurons oberved
- each code word: firing pattern of that neuron, digit 1 for each time intervals Δt that contained a spike and 0 otherwise
- zero-word baseline of no activity (for comparison)
- a neural code for N neurons is a sum $C_1 \vee \cdots \vee C_N$ with $C_i = \{c_0, c\}$ with zero-word c_0 and firing pattern c of i-th neuron

Category of weighted codes

- category of weighted binary codes $\mathcal{W}\mathrm{Codes}_{n,*}$
- objects pairs (C,ω) of a code C of length n containing zero-word c_0 and function $\omega : C \to \mathbb{R}$ assigning (signed) weight to each code word, with $\omega(c_0) = 0$
- morphisms $\phi = (f,\lambda) : (C,\omega) \to (C',\omega')$ with $f : C \to C'$ mapping the zero-word to itself and $f(\mathrm{supp}(\omega)) \subset \mathrm{supp}(\omega')$ and weights $\lambda_{c'}(c)$ for $c \in f^{-1}(c')$
- sum $(C,\omega) \oplus (C',\omega') = (C \vee C', \omega \vee \omega')$ with $\omega \vee \omega'|_C = \omega$ and $\omega \vee \omega'|_{C'} = \omega'$
- zero object $(\{c_0\}, 0)$

Equalizer: linear model

- a summing functor Φ in the equalizer of source and target functors

$$\Sigma^{eq}_{\mathcal{W}\mathrm{Codes}_{n,*}}(G) := \mathrm{eq}(s,t : \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_G) \rightrightarrows \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(V_G))$$

  is a summing functor $\Phi \in \Sigma_{\mathcal{W}\mathrm{Codes}_{n,*}}(E_G)$ with conservation laws $\Phi(s^{-1}(A)) = \Phi(t^{-1}(A))$ for $A \subset V_G$

- If directed graph G has a single outgoing edge at each vertex, $\{e \in E_G \,|\, s(e) = v\} = \{\mathrm{out}(v)\}$, then equalizer condition

$$(C_{\mathrm{out}(v)}, \omega_{\mathrm{out}(v)}) = \oplus_{t(e)=v}(C_e, \omega_e),$$

- can be seen as a kind of categorical version of linear neuron model

Discrete and continuous Hopfield dynamics

- discrete version (binary neurons)

$$\nu_j(n+1) = \begin{cases} 1 & \text{if } \sum_k T_{jk}\nu_k(n) + \eta_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

- continuous version (neuron firing rates as variables and threshold-linear dynamics)

$$\frac{dx_j}{dt} = -x_j + \left(\sum_k W_{jk} x_j + \theta_j \right)_+$$

W_{jk} real-valued connection strengths, θ_j constant external inputs, and $(\cdot)_+ = \max\{0, \cdot\}$ threshold function

- finite difference version

$$\frac{x_j(t+\Delta t) - x_j(t)}{\Delta t} = -x_j + (\sum_k W_{jk} x_k(t) + \theta_j)_+$$

(versions with or without "leak term" $-x_j$ on r.h.s.)

Categorical Hopfield dynamics: Step 1

- as above $\Sigma_{\mathcal{C}}^{eq}(G)$ for a network G and category of resources $\mathcal{C}$
- $\rho : \mathcal{C} \to \mathcal{R}$ functor to another category of resources (maybe same) with respect to which dynamics is measured
- $(R, +, \succeq)$ preordered semigroup of category $\mathcal{R}$
- will use relation $r_C \succeq 0$ for class of $\rho(C)$ for threshold-dynamics
- $\mathcal{E}(\mathcal{C}) = \mathrm{Func}(\mathcal{C}, \mathcal{C})$ category of monoidal endofunctors of $\mathcal{C}$, morphisms natural transformations
- sum of endofunctors defined pointwise $(F \oplus F')(C) = F(C) \oplus F'(C)$ for all $C \in \mathrm{Obj}(\mathcal{C})$.

Categorical Hopfield dynamics: Step 2

- *bisumming functors* $T : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathcal{E}(\mathcal{C})$ summing in both arguments
- *coordinates*: $T_{ee'}$ with $T_{A,B} = \oplus_{e\in A, e'\in B} T_{ee'}$
- $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E)$ category of bisumming functors with invertible natural transformations
- $\Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(G)$ equalizer of functors

$$s, t : \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(E) \rightrightarrows \Sigma^{(2)}_{\mathcal{E}(\mathcal{C})}(V)$$

Categorical Hopfield dynamics: Step 3

- initial condition $\Phi_0 \in \Sigma_{\mathcal{C}}^{eq}(G)$: set $X_A(0) := \Phi_0(A)$ (or just $X_e(0) := \Phi_0(e)$)
- fixed summing functor $\Psi \in \Sigma_{\mathcal{C}}^{eq}(G)$: set $\Theta_e = \Psi(e)$
- take $Y_e(n) := \oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e$
- $r_{Y_e(n)}$ the class in $(R, +, \succeq)$ of the object $\rho(Y_e(n))$ in $\mathcal{R}$
- threshold $(\cdot)_+$: $(Y_e(n))_+ = \oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e$ if $r_{Y_e(n)} \succeq 0$ and zero-object of $\mathcal{C}$ otherwise
- equation

$$X_e(n+1) = X_e(n) \oplus (\oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+$$

or variant $X_e(n+1) = (\oplus_{e' \in E} T_{ee'}(X_{e'}(n)) \oplus \Theta_e)_+$ (leaking term or not)

Some properties of the dynamics

- $X_A(n) =: \Phi_n(A)$ defines a sequence of summing functors in $\Sigma^{eq}_{\mathcal{C}}(G)$
- assignment $\mathcal{T} : \Phi_n \mapsto \Phi_{n+1}$ defined by solution defines endofunctor $\mathcal{T} : \Sigma^{eq}_{\mathcal{C}}(G) \to \Sigma^{eq}_{\mathcal{C}}(G)$
- induced discrete topological dynamical system τ on realization $|\mathcal{N}(\Sigma^{eq}_{\mathcal{C}}(G))| = B\Sigma^{eq}_{\mathcal{C}}(G)$
- for $\mathcal{C} = \mathcal{W}\mathrm{Codes}_{n,*}$ with a measuring semigroup, categorical Hopfield dynamics induces usual (finite difference) Hopfield dynamics on the weights
- Question: general results in categorical setting about existence of solutions and behavior?

Category of concurrent/distributed computing architectures

- category of transition systems
  - G. Winskel, M. Nielsen, *Categories in concurrency*, in "Semantics and logics of computation (Cambridge, 1995)", pp. 299–354, Publ. Newton Inst., 14, Cambridge Univ. Press, 1997.
- models of computation that involve parallel and distributed processing
- objects $\tau = (S, \iota, \mathcal{L}, \mathcal{T})$ with S set of possible states of the system, ι initial state, $\mathcal{L}$ set of labels, $\mathcal{T}$ set of possible transitions, $\mathcal{T} \subseteq S \times \mathcal{L} \times S$ (specified by initial state, label of transition, final state)
- directed graph with vertex set S and with set of labelled directed edges $\mathcal{T}$

- $\mathrm{Mor}_{\mathcal{C}}(\tau,\tau')$ of transition systems pairs (σ,λ), function $\sigma : S \to S'$ with $\sigma(\iota) = \iota'$ and (partially defined) function $\lambda : \mathcal{L} \to \mathcal{L}'$ of labeling sets such that, for any transition $s_{in} \stackrel{\ell}{\to} s_{out}$ in $\mathcal{T}$, if $\lambda(\ell) \in \mathcal{L}'$ is defined, then $\sigma(s_{in}) \stackrel{\lambda(\ell)}{\to} \sigma(s_{out})$ is a transition in $\mathcal{T}'$
- categorical sum

$$(S,\iota,\mathcal{L},\mathcal{T})\oplus(S',\iota',\mathcal{L}',\mathcal{T}') = (S\times\{\iota'\}\cup\{\iota\}\times S', (\iota,\iota'), \mathcal{L}\cup\mathcal{L}', \mathcal{T}\sqcup\mathcal{T}')$$

$$\mathcal{T}\sqcup\mathcal{T}' := \{(s_{in},\ell,s_{out})\in\mathcal{T}\}\cup\{(s'_{in},\ell',s'_{out})\in\mathcal{T}'\}$$

where both sets are seen as subsets of

$$(S\times\{\iota'\}\cup\{\iota\}\times S')\times(\mathcal{L}\cup\mathcal{L}')\times(S\times\{\iota'\}\cup\{\iota\}\times S')$$

- zero object is given by the stationary single state system $S = \{\iota\}$ with empty labels and transitions

Grafting

- $\tau_i = (S_i, \iota_i, \mathcal{L}_i, \mathcal{T}_i)$ for $i = 1, 2$ objects in category $\mathcal{C}$ of transition systems
- a choice of two states $s \in S_1$ and $s' \in S_2$
- grafting $\tau_{s,s'} = (S, \iota, \mathcal{L}, \mathcal{T})$ in $\mathcal{C}$ with $S = S_1 \cup S_2$, $\iota = \iota_1$, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{e\}$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{(s, e, s')\}$
- $\mathcal{C}' \subset \mathcal{C}$ subcategory of transition systems τ with a single final state $q \in S$
- then grafting $\tau_1 \star \tau_2$ given by τ_{q_1,ι_2} with final state of τ_1 grafted to initial state of τ_2
- G finite acyclic directed graph and ω a topological ordering vertex set V_G then given $\{\tau_v\}_{v \in V}$ objects of $\mathcal{C}'$ there is a well defined grafting $\tau_{G,\omega}$ of the τ_v that is also an object in $\mathcal{C}'$

Strongly connected components, condensation graph, and computational architecture functor

- finite directed graph G: subset $V' \subset V_G$ is a strongly connected component if each vertex in V' reachable through oriented path in G from any other
- condensation graph $\bar{G}$ is a directed acyclic graph: obtained from G by contracting each subgraph of a strongly connected component to a single vertex
- $\mathcal{G} := \mathrm{Func}(2, \mathcal{F})$ category of finite directed graphs
- $\Delta_{\mathcal{G}}$ category with objects pairs (G, Φ) with $G \in \mathrm{Obj}(\mathcal{G})$ and $\Phi \in \Sigma_{\mathcal{C}}(V_G)$, morphisms (α, α_*) with $\alpha : G \to G'$ and $\alpha_*(\Phi)(A) = \Phi(\alpha_V^{-1}(A))$
- $\Delta'_{\mathcal{G}}$ subcategory of $\Delta_{\mathcal{G}}$ with objects (G, Φ) where summing functor Φ takes values in $\mathcal{C}'$
- functor $\Xi_0 : \Delta'_{\mathcal{G}} \to \mathcal{C}'$ assigning to an object (G, Φ) the grafting $\tau_{\bar{G},\bar{\omega}}$ along the condensation graph $\bar{G}$ of the $\Phi(V_{G_i})$ with G_i the strongly connected components of G

Modeling computational architectures of neuronal networks

- local automata model (discretized) individual neurons with pre-synaptic and post-synaptic activity
- grafting of these automata where their inputs and outputs are connected model connectivity of the network
- can adapt this setting to model non-local neuromodulation effects (distributed computing models of neuromodulation: Potjans–Morrison–Diesmann)

Keeping track of associated measures of informational complexity

Information and codes

- probability distribution associated to neural codes through it firing rate
- word of length n recording a digit 1 for each time interval Δt that contains a spike and a 0 otherwise
- (Σ_2^+, μ_P) with $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ and μ_P Bernoulli measure
- $\mu_P(\Sigma_2^+(w_1, \ldots, w_n)) = p^{a_n(w)}(1-p)^{b_n(w)}$ with $a_n(w)$ number of 1's and $b_n(w) = n - a_n(w)$ the number of zeros
- for large n neural code C in Shannon Random Code Ensemble of (Σ_2^+, μ_P)

$$\lim_{n\to\infty} \frac{a_n(w)}{n} \overset{a.e.}{=} p$$

- category $\mathcal{P}_f$ of finite probabilities with fiberwise measures (not normalized) as morphisms
- $\phi = (f, \Lambda) : (X, P_X) \to (Y, P_Y)$ with $f : X \to Y$ pointed map $f(\mathrm{supp}(P_X)) \subset \mathrm{supp}(P_Y)$ and $\Lambda = \{\lambda_y\}$ on fibers $f^{-1}(y) \subset X$, with $\lambda_{y_0}(x_0) > 0$ and $P_X(A) = \sum_{y \in Y} \lambda_y(A \cap f^{-1}(y)) \, P_Y(y)$
- $\mathcal{P}_f$ has zero object and sum
- functor $P : \mathrm{Codes}_{n,*} \to \mathcal{P}_f$

$$P_C(c) = \begin{cases} \frac{b(c)}{n(\#C-1)} & c \neq c_0 \\ 1 - \sum_{c' \neq c_0} \frac{b(c')}{n(\#C-1)} & c = c_0 \end{cases}$$

- consistent assignments of codes to a network $\Rightarrow$ assignment of probabilities
- information: $\mathcal{P}_{f,s}$ subcategory with $f : X \to Y$ surjections and $\lambda_y(x)$ for $x \in f^{-1}(y)$ probability measures, then Shannon entropy is a functor $S : \mathcal{P}_{f,s} \to \mathbb{R}$ (with $(\mathbb{R}, \geq)$ thin category)

Integrated Information (Tononi)

1. G. Tononi G (2008) *Consciousness as integrated information: A provisional manifesto*, Biol. Bull. 215 (2008) N.3, 216–242.
2. M. Oizumi, N. Tsuchiya, S. Amari, *Unified framework for information integration based on information geometry*, PNAS, Vol. 113 (2016) N. 51, 14817–14822.

- want to measure amount of informational complexity in a system that is not separately reducible to its individual parts
- possibilities of causal relatedness among different parts of the system

Computing integrated information

- consider all possible ways of splitting a given system into subsystems
- the state of the system at a given time t is described by a set of observables X_t and the state at a near-future time by X_{t+1}
- partition λ into N subsystems $\Rightarrow$ splitting of these variables $X_t = \{X_{t,1}, \ldots, X_{t,N}\}$ and $X_{t+1} = \{X_{t+1,1}, \ldots, X_{t+1,N}\}$ into variables describing the subsystems
- all causal relations among the $X_{t,i}$ or among the $X_{t+1,i}$, also causal influence of the $X_{t,i}$ on the $X_{t+1,j}$ through time evolution captured (statistically) by joint probability distribution $\mathbb{P}(X_{t+1}, X_t)$
- compare information content of this joint distribution with distribution where only causal dependencies between X_{t+1} and X_t through evolution within separate subsystem not across subsystems

- set $\mathcal{M}_\lambda$ of probability distributions $\mathbb{Q}(X_{t+1}, X_t)$ with property that

$$\mathbb{Q}(X_{t+1,i}|X_t) = \mathbb{Q}(X_{t+1,i}|X_{t,i})$$

  for each subset $i = 1, \ldots, N$ of the partition λ
- minimize Kullback-Leibler divergence between actual system and its best approximation in $\mathcal{M}_\lambda$ over choice of partition λ
- integrated information

$$\Phi = \min_\lambda \min_{\mathbb{Q}\in\mathcal{M}_\lambda} \mathrm{KL}(\mathbb{P}(X_{t+1}, X_t)||\mathbb{Q}(X_{t+1}, X_t))$$

- value Φ represents additional information in the whole system not reducible to smaller parts

Cohomological view of information (Bennequin, Badot, Vigneaux)

- *abelian* category describing probability data: category $\mathcal{IS}$ of finite information structures with random variables and simplicial set of associated probabilities, with functor to vector spaces: real valued measurable functions; resulting abelian category of modules over a sheaf of algebras
- Hochschild cochain complex and associated cohomology
- Shannon entropy, KL divergence, Tsallis entropy: all have interpretation as nontrivial 1-homology generators

Use this setting to construct:

- contravariant functor $\mathcal{I} : \mathrm{Codes}_{n,*} \to \mathcal{IS}$
- using above construction functor from $\Sigma^{eq}_{\mathrm{Codes}_{n,*}}(G)$ to cochain complexes and cohomology
- using Hochschild cocycle interpretation of KL divergence obtain cohomological interpretation for integrated information, with functorial map from $\Sigma^{eq}_{\mathrm{Codes}_{n,*}}(G)$

Further steps

- neural codes generate homotopy types, in the form of the nerve simplicial set of an open covering associated to a (convex) code (Curto et al.)
- recover that homotopy type from the above setting with information structures
- combine the simplicial sets obtained in this way with those obtained via Gamma spaces describing assignments of resources to network
- simplicial sets $K(G)$ given by clique complex of network G also realized as special case finite information structures

Conclusion: proposed view

- working hypothesis: the brain *represents* the stimulus space through a *homotopy type*
- mathematical modeling of network architectures in the brain should include mechanisms that generates homotopy types (Gamma spaces, information structures)
- higher topological complexity in these homotopy types implies (but is not implies by) higher values of (cohomological) integrated information
- Question: is there a good model of a "qualia" in terms of homotopy types?