Non-Archimedean Holography

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Physical Mathematics
in honor of Yuri Manin

This talk is based on:

- Matthew Heydeman, Matilde Marcolli, Ingmar Saberi, Bogdan Stoica, *Tensor networks, p-adic fields, and algebraic curves: arithmetic and the* AdS_3/CFT_2 *correspondence*, arXiv:1605.07639
- Steven S. Gubser, Matthew Heydeman, Christian Jepsen, Matilde Marcolli, Sarthak Parikh, Ingmar Saberi, Bogdan Stoica, Brian Trundy, *Edge length dynamics on graphs with applications to p-adic AdS/CFT*, arXiv:1612.09580
- additional work in progress

Manin's *Arithmetical Physics*

- Yuri I. Manin, *Reflections on Arithmetical Physics*, pp. 293–303, Perspectives in Physics, Academic Press, 1989.

• Observation: Polyakov measure for bosonic string and the Faltings height function at algebraic points of the moduli space of curves ... Is there an adelic Polyakov measure? An arithmetic expression for the string partition function?

• General Questions: Are the fundamental laws of physics *adelic*? Does physics in the Archimedean setting (partition functions, action functionals, real and complex variables) have p-adic shadows? Do these provide convenient "discretized models" of physics powerful enough to recover the Archimedean counterpart?

AdS/CFT Holographic Correspondence

- bulk/boundary spaces
- hyperbolic geometry in the bulk (Lorentzian AdS spaces, Euclidean hyperbolic spaces $\mathbb{H}^{d+1}$)
- conformal boundary at infinity:
  $\partial\mathbb{H}^3 = \mathbb{P}^1(\mathbb{C})$ (AdS_3/CFT_2) or
  $\partial\mathbb{H}^2 = \mathbb{P}^1(\mathbb{R})$ (AdS_2/CFT_1)
- AdS/CFT correspondence: a d-dimensional conformal field theory on the boundary related to a gravitational theory on the $d+1$ dimensional bulk

AdS/CFT Holography developed in String Theory since the 1990s

- E. Witten, *Anti-de Sitter space and holography*, arXiv:hep-th/9802150

Schottky Groups and Holography

- Yuri I. Manin, 3-*dimensional hyperbolic geometry as* ∞-*adic Arakelov geometry'*, Invent.Math. 104 (1991) N.2, 223–243
- Yuri I. Manin, M. Marcolli, *Holography principle and arithmetic of algebraic curves*, Adv. Theor. Math. Phys. 5 (2001), no. 3, 617–650.

• Holography on Riemann surfaces:

- Conformal boundary: $X(\mathbb{C})$ Riemann surface genus g
  Schottky uniformization $X(\mathbb{C}) = \Omega_\Gamma/\Gamma$ with $\Gamma \sim \mathbb{Z}^{\star g}$
  $\Omega_\Gamma = \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$ domain of discontinuity (Λ_Γ limit set)
- Bulk space: hyperbolic handlebody $\mathfrak{X}_\Gamma = \mathbb{H}^3/\Gamma$ with $X(\mathbb{C}) = \partial \mathfrak{X}_\Gamma$
- Green function on $X(\mathbb{C})$ in terms of geodesic lengths in the bulk space $\mathfrak{X}_\Gamma$

- Green function on X with Schottky uniformization:

$$g((a)-(b),(c)-(d)) = \sum_{h\in\Gamma} \log|\langle a,b,hc,hd\rangle|$$

$$-\sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h\in S(g_\ell)} \log|\langle z^+(h), z^-(h), c, d\rangle|$$

$S(\gamma)$ conjugacy class of γ in Γ

$\langle a,b,c,d\rangle$ = cross ratio

- In terms of geodesics in the handlebody $\mathfrak{X}_\Gamma$

$$-\sum_{h\in\Gamma} \mathrm{ordist}(a*\{hc,hd\}, b*\{hc,hd\})$$

$$+\sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h\in S(g_\ell)} \mathrm{ordist}(z^+(h)*\{c,d\}, z^-(h)*\{c,d\}).$$

Coefficients $X_\ell(a,b)$ also in terms of geodesics

Example: Bañados–Teitelboim–Zanelli black hole (Euclidean)

• Genus one: $\mathfrak{X}_q = \mathbb{H}^3/(q^{\mathbb{Z}})$ and $X_q(\mathbb{C}) = \mathbb{C}^*/(q^{\mathbb{Z}})$ (Jacobi uniformization); action: $q : (z, y) \mapsto (qz, |q|y)$

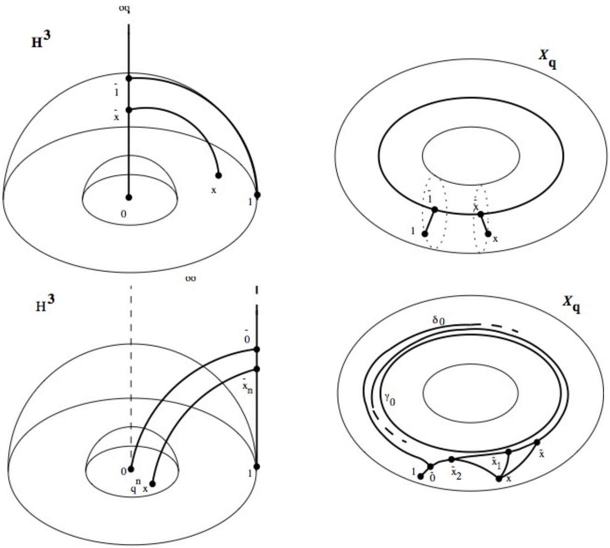
• mass and angular momentum of black hole:

$$q = \exp\left(\frac{2\pi(i|r_-| - r_+)}{\ell}\right) \quad r_\pm^2 = \frac{1}{2}\left(M\ell \pm \sqrt{M^2\ell^2 + J}\right)$$

with $-1/\ell^2 =$ cosmological constant

• higher genus Euclidean black holes bulk $\mathfrak{X}_\Gamma$ boundary genus g Riemann surface $X(\mathbb{C})$

- Kirill Krasnov, *Holography and Riemann surfaces*, Adv. Theor. Math. Phys. 4 (2000) no. 4, 929–979.



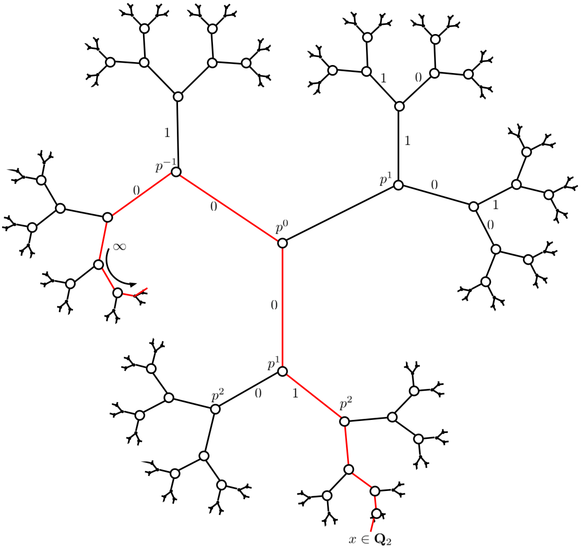
Bañados–Teitelboim–Zanelli black hole (Euclidean)

p-adic version of AdS/CFT Holography

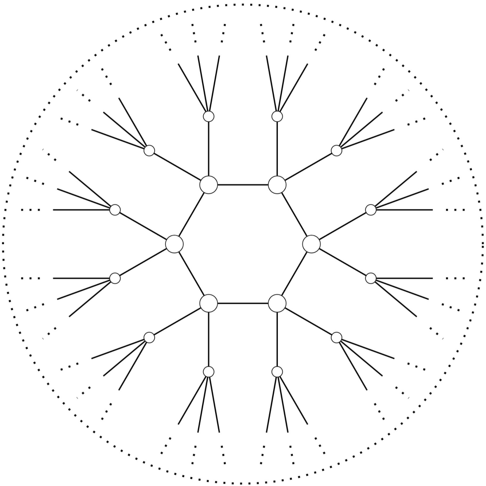
- $\mathbb{K}$ finite extension of $\mathbb{Q}_p$
- bulk space $\Delta_{\mathbb{K}}$ Bruhat-Tits building; boundary $\partial\Delta_{\mathbb{K}} = \mathbb{P}^1(\mathbb{K})$
- p-adic Schottky groups $\Gamma \subset \mathrm{PGL}(2, \mathbb{K})$
- Schottky–Mumford curve boundary $X_\Gamma = \Omega_\Gamma/\Gamma$
- bulk: graph $\Delta_{\mathbb{K}}/\Gamma$; central finite graph $G = \Delta_\Gamma/\Gamma$
  $\Delta_\Gamma \subset \Delta_{\mathbb{K}}$ tree spanned by axes of hyperbolic $\gamma \in \Gamma$
- G dual graph of closed fiber of min model over $\mathcal{O}_{\mathbb{K}}$

Analogous result on geodesics on the bulk and correlation functions on the boundary based on

- Yu.I. Manin, V. Drinfeld, *Periods of p-adic Schottky groups*, J. Reine u. Angew. Math., vol. 262–263 (1973) 239–247



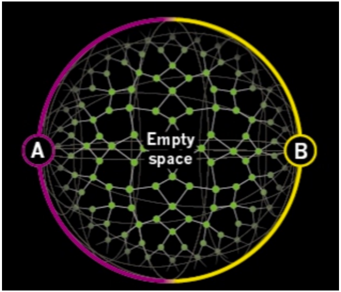
Bruhat–Tits tree of $\mathbb{K} = \mathbb{Q}_2$ and geodesics



p-adic Bañados–Teitelboim–Zanelli black hole with $\mathbb{K} = \mathbb{Q}_3$

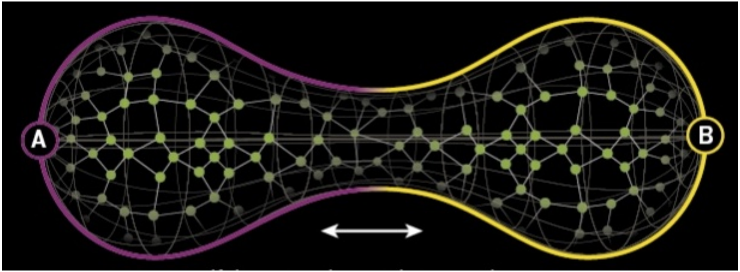
More Recent Developments in AdS/CFT: Quantum Information

- Instead of correlation functions of a CFT on the boundary matched with geodesics (classical gravity) on the bulk, focus on Information (Entanglement Entropy) of quantum states on the boundary and geometry (classical gravity) on the bulk.



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Entanglement between quantum fields in regions A and B decreases when corresponding regions of bulk space are pulled apart: dynamics of spacetime geometry (= gravity) constructed from quantum entanglement



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Ryu–Takayanagi Formula:

Entanglement Entropy and Bulk Geometry

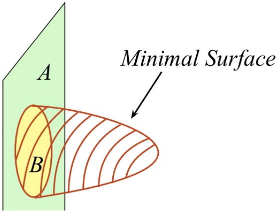
- Entanglement Entropy: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\Psi\rangle\langle\Psi|), \quad \mathcal{S}_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

- Entanglement and Geometry: (conjecture)

$$\mathcal{S}_A = \frac{\mathcal{A}(\Sigma_{\min})}{4G}$$

area of minimal surface in the bulk with given boundary $\partial A = \partial B$



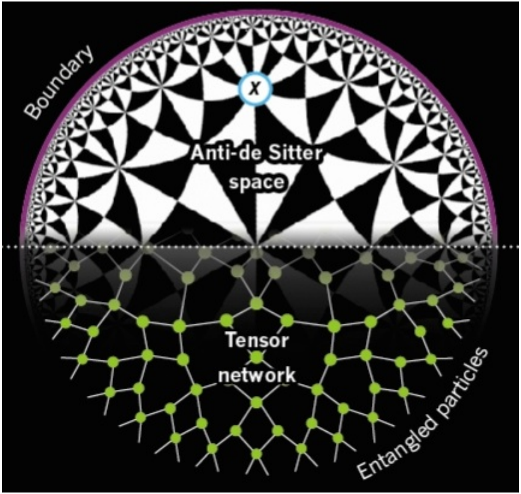
from T.Nishioka,S.Ryu,T.Takayanagi, "Holographic entanglement entropy: an overview", J.Phys.A 42 (2009) N.50, 504008

Tensor Networks, Quantum Codes, and Geometry from Information

- Fernando Pastawski, Beni Yoshida, Daniel Harlow, John Preskill, *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, JHEP 06 (2015) 149

Main Idea: Bulk spacetime geometry is the result of *entanglement* of quantum states in the boundary through a network of quantum error correcting codes

- quantum codes by perfect tensors: maximal entanglement across bipartitions
- network of perfect tensors with contracted legs along a tessellation of hyperbolic space
- uncontracted legs at the boundary (physical spins), and at the center of each tile in the bulk (logical spins)
- holographic state: pure state of boundary spins
- logical inputs on the bulk: encoding by the tensor network (holographic code)



from R.Cowen, "The quantum source of space-time", Nature 527 (2015) 290–293

Pentagon tile holographic code [PYHP]

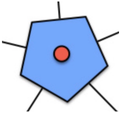
• perfect tensors: $T_{i_1,\dots,i_n}$ such that, for $\{1,\dots,n\} = A \cup A^c$ with $\# A \leq \# A^c$, isometry $T : \mathcal{H}_A \to \mathcal{H}_{A^c}$; perfect code (encodes one qbit to $n-1$)

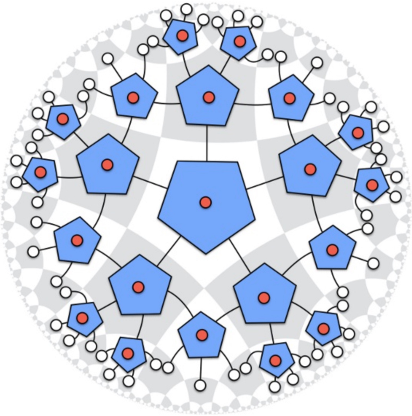
• six legs perfect tensor $T_{i_1\dots,i_6}$: five qbit perfect code $[[5,1,3]]_2$-quantum code:

$$\mathcal{C} \subset \mathcal{H}^{\otimes 5}, \quad \mathcal{C} = \{\psi \in \mathcal{H}^{\otimes 5} \,:\, S_j \psi = \psi\}$$

$$S_1 = X \otimes Z \otimes Z \otimes X \otimes I$$

X, Y, Z Pauli gates and $S_2, S_3, S_4, S_5 = S_1 S_2 S_3 S_4$ cyclic perms, with $\mathcal{H} = \mathbb{C}^2$ one qbit Hilbert space





from F.Pastawski, B.Yoshida, D.Harlow, J.Preskill, *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, JHEP 06 (2015) 149

Discretization of the AdS/CFT Correspondence

• Can a *p*-adic AdS/CFT correspondence deliver a discretized version of Holography, compatible with the Archimedean AdS_3/CFT_2 (complex) and AdS_2/CFT_1 (real) versions and with the tensor networks idea?

• What is *p*-adic AdS/CFT?

- at the level of boundary field theory?
- at the level of bulk gravity?
- in terms of tensor networks and holographic quantum codes?

• is there an *adelic* AdS/CFT that combines *p*-adic and Archimedean?

p-adic AdS/CFT: boundary field theory

- Literature on p-adic CFT
  - L. O. Chekhov, A. D. Mironov, and A. V. Zabrodin, *Multiloop calculations in p-adic string theory and Bruhat– Tits Trees*, Communications in Mathematical Physics 125, (1989), pp. 675–711.
  - E. Melzer, *Nonarchimedean conformal field theories*, Int. J. of Modern Physics A 4, no. 18 (1989), 4877–4908.
- important differences with respect to Archimedean:
  - global $\mathrm{PGL}_2(\mathbb{Q}_p)$-symmetries (primaries)
  - no local conformal algebra (no descendants)
  - correlation function between two primary fields inserted at x and y (scaling dimension Δ_n)

$$\langle \phi_m(x)\phi_n(y)\rangle = \frac{\delta_{n,m}}{|x-y|_p^{2\Delta_n}}$$

  - because ultrametric: 3-point and 4-point functions determined *exactly* by operator product expansion (OPE) coefficients

Classical (complex valued) p-adic fields

- mode expansion (Archimedean)

$$\phi(x) = \int_{\mathbb{R}} e^{2ikx} \hat{\phi}(k)\, dx$$

- $\chi_k(x) = e^{2\pi i\{kx\}}$ additive characters of $\mathbb{Q}_p$ with $\{\cdot\} : \mathbb{Q}_p \to \mathbb{Q}$ fractional part truncation

$$\{\sum_{k=m}^{\infty} a_k p^k\} = \sum_{k=m}^{-1} a_k p^k$$

- mode expansion (p-adic) with Haar measure on $\mathbb{Q}_p$

$$\phi(x) = \int_{\mathbb{Q}_p} e^{2i\{kx\}} \hat{\phi}(k)\, d\mu(x)$$

- non-local Valdimirov derivative

$$\partial_p^s f(x) = \int_{\mathbb{Q}_p} \frac{f(x') - f(x)}{|x' - x|_p^{s+1}} d\mu x'$$

like a Cauchy formula for derivatives as contour integrals

quantizing classical fields

- quadratic action for a scalar field (first Vladimirov derivative)

$$S_p[\phi] = -\int_{\mathbb{Q}_p} \phi(s)\partial_p\phi(x)\, d\mu(x)$$

- conformal symmetry: conformal dimension Δ_n

$$\phi_n(x) \mapsto |cx+d|_p^{2\Delta_n}\phi_n(x)$$

- free boson: dim $\Delta = 0$; Vladimirov derivative $\partial_p\phi(x)$ weight $|cx+d|_p^2$, conformal dim 1
- functional integral: $\mathbb{C}$-valued fields so usual form $Z_p = \int \mathcal{D}\phi\, e^{-S_p[\phi]}$

$$Z_p[J] = \int \mathcal{D}\phi\, \exp\left(-S_p[\phi] + \int_{\mathbb{Q}_p} J(x')\phi(x')dx'\right)$$

Green functions

- Green functions for Vladimirov derivative $\partial_{(p)} G(x-y) = -\delta(x-y)$
- Momentum space: $\widetilde{G}(k) = -\frac{\chi(ky)}{|k|_p}$

$$G(x-y) = -\int_{\mathbb{Q}_p} \frac{\chi(k(y-x))}{|k|_p} dk = -\int_{\mathbb{Q}_p} \frac{\chi(ku)}{|k|_p} dk$$

- Regularization at $k \to 0$ and p-adic Gamma function:

$$\lim_{\alpha \to 0} \int_{\mathbb{Q}_p} \chi(ku)|k|_p^{\alpha-1} dk = \lim_{\alpha \to 0} \Gamma_p(\alpha)|u|_p^{-\alpha}$$

- obtain 2-point function behavior (with $a \to 0$)

$$\langle 0|\phi(x)\phi(y)|0\rangle \sim \log \left|\frac{x-y}{a}\right|_p$$

scalar fields on the bulk Bruhat–Tits tree $\mathcal{T} = \mathcal{T}_{\mathbb{Q}_p}$

- $\phi(v)$ over vertices $C^0(\mathcal{T})$; $\psi(e)$ over edges $C^1(\mathcal{T})$

$$\begin{array}{ll} d : C^0(\mathcal{T}) \to C^1(\mathcal{T}), & (d\phi)(e) = \phi(t_e) - \phi(s_e) \\ d^\dagger : C^1(\mathcal{T}) \to C^0(\mathcal{T}), & (d^\dagger\psi)(v) = \sum_e \pm\psi(e) \end{array}$$

- Laplacian $\Delta\phi(v) = \sum_{d(v,v')=1} \phi(v') - (p+1)\phi(v)$ and $\Delta\psi(e) = \sum_{e'} \pm\psi(e') - 2\psi(e)$ (sum over $2p$ adjacent edges)
- massless quadratic action $S[\phi] = \sum_e |d\phi(e)|^2$
- wave equation $\Delta\psi = 0$ and massive $(\Delta - m^2)\psi = 0$ (Zabrodin)
- plane waves $\epsilon_{\kappa,x}(v) = p^{-\kappa d_x(v)}$ with $d_x(v_0) = 0$ at root vertex and $d_x(v) \to -\infty$ as $v \to x$ boundary point

$$\Delta\epsilon_{\kappa,x} = m_\kappa^2 \epsilon_{\kappa,x} = ((p^\kappa + p^{1-\kappa}) - (p+1))\epsilon_{\kappa,x}$$

- when $\Re(\kappa) > 0$ plane wave solution $\to 0$ on boundary except at x (where divergent)
- for κ real, min of m_κ^2 at $\kappa = 1/2$: bound on mass of AdS fields

$$m_\kappa^2 \geq -(\sqrt{p} - 1)^2$$

bulk fields reconstruction from boundary field theory

- bulk harmonic functions

$$\phi(v) = \frac{p}{p+1} \int d\mu_0(x)\, \phi_0(x)\, \varepsilon_{1,x}(v)$$

$d\mu_0(x)$ is Patterson-Sullivan measure on $\mathbb{P}^1(\mathbb{Q}_p)$

- $\delta(a \to b, c \to d)$ overlap (with sign) of the two oriented paths in the tree

$$\langle v, x \rangle = \delta(v_0 \to v, v_0 \to x) + \delta(v \to x, v_0 \to v)$$

- ball B_w determined by rays from vertex w

$$\phi_w(v) = \int_{\partial B_w} d\mu_0(x)\, p^{\kappa \langle v, x \rangle}$$

- cases $v \notin B_w$ and $v \in B_w$ (with $x \in B_v$ or $x \notin B_w$) sum to

$$\phi_w(v) = \left(\frac{p^{-2\kappa} - 1}{p^{1-2\kappa} - 1} \right) p^{(\kappa-1) d(v_0, v)} + \frac{p-1}{p} \left(\frac{p^{(2\kappa-1) d(v_0, w)}}{p^{2\kappa-1} - 1} \right) p^{-\kappa\, d(v_0, v)}$$

bulk reconstruction: massive fields

- behavior $\phi(v) \sim p^{(\kappa-1)d(v_0,v)}\phi_0(x)$ as $v \to x$

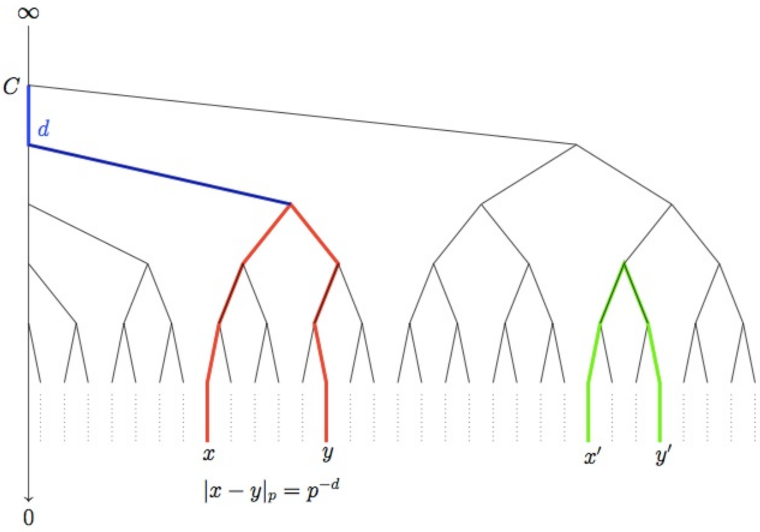
$$\phi(v) = \frac{p^{1-2\kappa}-1}{p^{-2\kappa}-1} \int d\mu_0(x)\phi_0(x)p^{\kappa\langle v,x\rangle}$$

- near the boundary $\langle v, x\rangle = -d(v_0, v) + 2\mathrm{ord}_p(x-y)$

$$\phi(v) = \left(\frac{p^{1-2\kappa}-1}{p^{-2\kappa}-1}\right) p^{-\kappa\, d(v_0,v)} \int d\mu(x) \frac{\phi_0(x)}{|x-y|_p^{2\kappa}}$$

- Vladimirov derivative as a "normal" derivative on the boundary: rate of change in holographic direction of reconstructed bulk function

$$\lim_{v\to y}(\phi(v)-\phi(y))p^{\kappa d(v_0,v)} = \left(\frac{p^{1-2\kappa}-1}{p^{-2\kappa}-1}\right) \partial_{(p)}^{2\kappa-1}\phi_0(y)$$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$|x-y|_p = p^{-d}$
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Holographic correspondence

- formulation as in Archimedean case of
  - E. Witten, *Anti-de Sitter space and holography*, arXiv:hep-th/9802150
  - classical scalar fields in a non-dynamical AdS background (fixed equal lengths of all tree edges)
  - relate asymptotics (and mass) of bulk fields to conformal dimension of boundary operators
  - existence and uniqueness solution of generalized Dirichlet problem for bulk equations of motion with given boundary conditions

- Ryu–Takayanagi formula (conjectural) entanglement entropy proportional to regularized length of infinite geodesic

$$d(x,y) = \lim_{\epsilon\to 0} \frac{2}{\log p} \log \left| \frac{x-y}{\epsilon} \right|_p$$

cutting off tree distance a from center v_0 with $a = \mathrm{ord}_p(\epsilon)$ with $d_a(x,y) = 2a + \frac{2}{\log p} \log |x-y|_p$

p-adic AdS/CFT: bulk gravity

- Combinatorial curvatures on simpicial complexes
- Combinatorial Ricci and scalar curvatures
- Dynamics of edge lengths and curvatures on graphs
- Nonisotropic solutions

Combinatorial Laplacian on simplicial complexes

- X locally finite simplicial complex dimension D
- $\mathcal{C}^k(X)$ abelian group compactly supported k-cochains of X
- bilinear forms $\langle \cdot, \cdot \rangle_k : \mathcal{C}^k(X) \times \mathcal{C}^k(X) \to \mathbb{R}$

$$\langle f_1, f_2 \rangle_k = \sum_{\sigma \,:\, \dim \sigma = k} \omega(\sigma)\, f_1(\sigma)\, f_2(\sigma)$$

  assigned weights $\omega(\sigma) \in \mathbb{R}^*_+$
- operators $d : \mathcal{C}^k(X) \to \mathcal{C}^{k-1}(X)$ and $\delta : \mathcal{C}^k(X) \to \mathcal{C}^{k+1}(X)$

$$df(\sigma) = \sum_\gamma \epsilon(\gamma, \sigma)\, f(\gamma)$$

$$\delta f(\sigma) = \sum_\tau \epsilon(\sigma, \tau)\, \frac{\omega(\tau)}{\omega(\sigma)} f(\tau)$$

  ϵ-signs relative orientations
- combinatorial Laplacians $\Delta_k : \mathcal{C}^k(X) \to \mathcal{C}^k(X)$ with $\Delta = \delta d + d\delta$

Combinatorial Weitzenböck formula (Forman)

- decomposing symmetric matrix $A = B(A) + F(A)$ non-negative definite $B(A)_{ij} = A_{ij}$ for $i \neq j$ and $B(A)_{ii} = \sum_{j\neq i} |A_{ij}|$ and diagonal $F(A)_{ii} = A_{ii} - \sum_{j\neq i} |A_{ij}|$
- apply to Laplacian Δ_k as matrix in o.n. basis (wrt bilinear form)
- for $\sigma = e$ (edges) Weitzenböck curvature $F_1(e)$: combinatorial Ricci curvature
- for vertices Weitzenböck curvature $F_0(v)$: combinatorial scalar curvature
- $F_1(e)$ involves adjacent vertices and faces (no faces in tree/graph case)
- $F_0(v)$ involves adjacent edges

Curvatures on Bruhat–Tits trees

$$F_1(e) = \mathrm{Ric}(e) = \omega(s(e)) \left(1 - \sum_{i=1}^{q} \sqrt{\frac{\omega(e)}{\omega(e_i)}} \right) + \omega(t(e)) \left(1 - \sum_{j=1}^{q} \sqrt{\frac{\omega(e)}{\omega(e_j)}} \right)$$

$$F_0(v) = R(v) = \omega(v)^2 \sum_{i=1}^{q+1} \frac{1}{\omega(e_i)} \left(1 - \sqrt{\frac{\omega(v_i)}{\omega(v)}} \right)$$

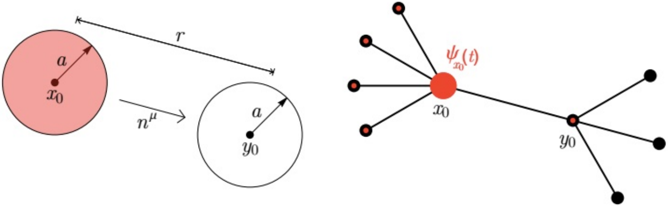
- naive version of Einstein equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$

$$\mathrm{Ric}(e) - \frac{1}{2} (R(s(e)) R(t(e)))^{1/2} \omega(e) + \Lambda \omega(e) = 0$$

constant metric $\omega(v) = \omega$ and $\omega(e) = \omega'$ has $F_0(v) \equiv 0$ and $\mathrm{Ric}(e) = \omega 2(1 - 2q)$; solutions with non-zero cosmological constant $\Lambda = \frac{\omega}{\omega'} 2(1 - 2q)$

- better approach: natural action functionals for edge lengths

- Y. Lin, L. Lu, and S.-T. Yau, *Ricci curvature of graphs*, Tohoku Math. J. (2) 63 (2011), no. 4 605–627



Ricci without Riemann tensor in terms of transport distance (Wasserstein distance of measures) between nearby balls: for graphs probability distribution $\psi_{x_0}(t)$ (small t)

$$\psi_x(t) = \begin{cases} 1 - \frac{d_J(x_0)}{D_{x_0}} t & x = x_0 \\ \frac{J_{x_0 x}}{D_{x_0}} t & x \sim x_0 \\ 0 & \text{otherwise} \end{cases}$$

D_{x_0} lapse function and $d_J(x_0) = \sum_{x \sim x_0} J_{x_0 x}$ normalization factor

Ricci curvature on a graph: edge lengths $a_e = a_{xy}$

$$\kappa_{xy} = \frac{1}{D_x a_{xy}} \left(\frac{1}{a_{xy}} - \sum_i \frac{1}{a_{xx_i}} \right) + \frac{1}{D_y a_{xy}} \left(\frac{1}{a_{xy}} - \sum_i \frac{1}{a_{yy_i}} \right) = \kappa_{x\to y} + \kappa_{y\to x}$$

- directed half of Ricci curvature: with $c_J(x) = \sum_{y\sim x} \sqrt{J_{xy}}$

$$\kappa_{x\to y} = \frac{\sqrt{J_{xy}}}{d_J(x)} \left(2\sqrt{J_{xy}} - c_J(x) \right)$$

- uniform tree of valence $q+1$ with all edges of equal length $a_e = a$ and $D_x = D$, get negatively curved

$$\kappa_{xy} = -\frac{2}{Da^2}(q-1)$$

- with $D_x = d_J(x)$ and uniform lengths $D = (q+1)/a^2$ and

$$\kappa_{xy} = -2\frac{q-1}{q+1}$$

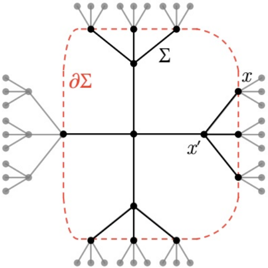
independent of scale a

Action functional for gravity

$$S = \sum_{x,y} (\kappa_{xy} - 2\Lambda)$$

sum over adjacent vertices, with $D_x = d_J(x)$

- need cutoff on size of graph (finite large graph) and Gibbons–Hawking type boundary term for gravity action (so variation in interior region with fixed boundary condition)
- require that for each vertex x on boundary $\partial\Sigma$ only one nearby vertex $x' \in \Sigma$, not on boundary



Variational Problem: Discrete Einstein Equation for trees

$$S_\Sigma = \sum_{x,y\in\Sigma} (\kappa_{xy} - 2\Lambda) + \sum_{x\in\partial\Sigma} k_x$$

$$k_x = k_0 + \sum_{y\sim x, y\neq x'} \kappa_{x\to y}$$

- edge lengths $a_e = a_{xy}$ and $J_e = J_{xy} = a_{xy}^{-2}$ "bond strength"
- $J_{xy} = 1 + j_{xy}$ perturb around uniform metric
- $c_J(x) = \sum_{y\sim x} \sqrt{J_{xy}}$ and $d_J(x) = \sum_{y\sim x} J_{xy}$
- $\gamma_{x\to y} = \sqrt{J_{xy}}\frac{c_J(x)^2}{d_J(x)^2} - \frac{c_J(x)}{d_J(x)}$
- discrete Einstein equations $\gamma_{xy} = \gamma_{x\to y} + \gamma_{y\to x} = 0$ (directed halves of variation of edge length action)
- constant solutions $J_{xy} = J$ constant for all edges
- use Λ and k_0 to regularize the action so finite in limit of infinite tree

$$\Lambda = -\frac{1}{3}\frac{q-1}{q+1}, \quad k_0 = \frac{q}{3}\frac{3q+1}{q+1}$$

Anisotropic solutions

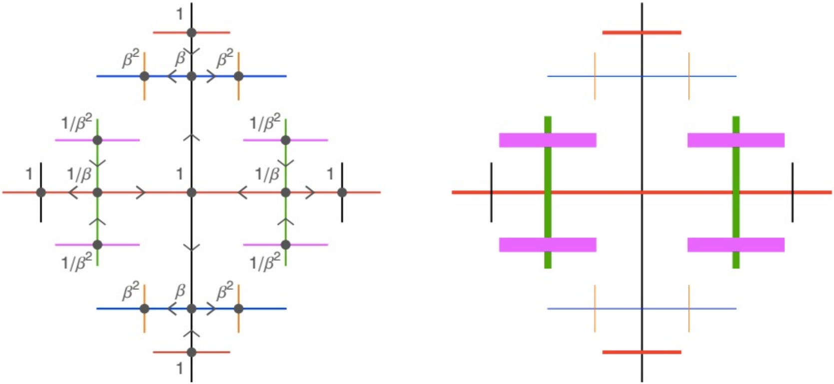
- Rewrite discrete Einstein equation as

$$\left(\lambda_{x\to y}-\frac{1}{2}\right)^2+\left(\lambda_{y\to x}-\frac{1}{2}\right)^2=\frac{1}{2}$$

$$\lambda_{x\to y}=\frac{1}{2\sigma_{x\to y}}:=\frac{\sqrt{J_{xy}}\,c_J(x)}{d_J(x)}$$

$$\lambda_{x\to y}=\frac{1}{2}+\frac{1}{\sqrt{2}}\cos\theta_{xy}, \quad \lambda_{y\to x}=\frac{1}{2}+\frac{1}{\sqrt{2}}\sin\theta_{xy}$$

- case all $\theta_{x\to y}=\frac{\pi}{4}$ recovers constant solution
- different angles α and $\tilde{\alpha}=\frac{\pi}{2}-\alpha$ and $\theta_{x\to x_i}=\alpha$ for i even and $\tilde{\alpha}$ for i odd get anisotropic solutions with J_{xy} determined in terms of $\beta=(\tilde{\sigma}/\sigma)^2$



1
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β^2
1

p-adic AdS/CFT: holographic codes

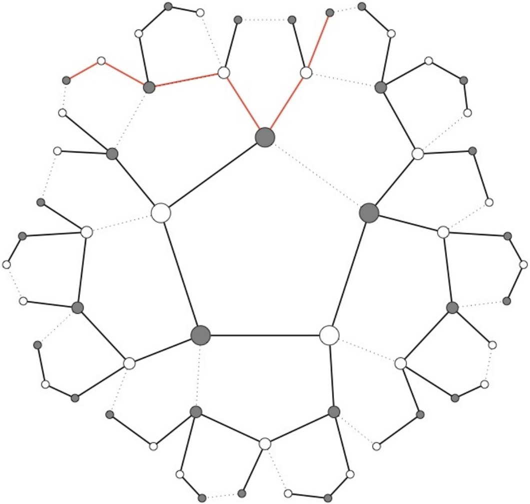
- Can simulate the holographic pentagon code using codes on a uniform tree?
- What kind of codes (classical and quantum) can be naturally built on a Bruhat-Tits tree?
- If replace the discretized bulk space (Bruhat-Tits tree) by the Drinfeld p-adic upper half plane

$$\Omega = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p)$$

  what kind of holographic codes can be constructed there?
- Relation to codes on Bruhat-Tits trees through the projection map $\Upsilon : \Omega \to \mathcal{T}_{\mathbb{Q}_p}$?

Holographic codes on $\mathbb{H}^2$ via trees

- Answer to first question is positive... but it requires a choice of planar embedding of the tree adapted to hyperbolic pentagon tiling



- perfect tensor codes on the tree $\mathcal{T}_{\mathbb{K}}$
  - single 3-ary qubit (qutritt) encodes to three 3-ary qubits

$$\begin{array}{rcl} |0\rangle & \mapsto & |000\rangle + |111\rangle + |222\rangle \\ |1\rangle & \mapsto & |012\rangle + |120\rangle + |201\rangle \\ |2\rangle & \mapsto & |021\rangle + |102\rangle + |210\rangle \end{array}$$

  - polynomial codes $f_a(x) = ax^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0$

$$|a\rangle \mapsto \sum_{b\in\mathbb{F}_q^d} \left(\otimes_{x\in\mathbb{F}_q} |f_a(x)\rangle \right)$$

  - example: $q = 5$

$$|a\rangle \mapsto \sum_{b_0, b_1\in\mathbb{F}_5} |b_0, b_0+b_1+a, b_0+2b_1+4a, b_0+3b_1+4a, b_0+4b_1+a\rangle$$

  - perfect tensors $T_{i_0\dots i_q}$ with $q+1$-legs for $\mathbb{K}$ finite extension of $\mathbb{Q}_p$ with $\mathbb{F}_q$ residue field

Classical and quantum codes on $\mathcal{T}_{\mathbb{K}}$

- fix a projective coordinate on $\mathbb{P}^1(\mathbb{K})$ determines root vertex v_0
- vertex corresponds to mod $\mathfrak{m}$ reduction $\mathbb{P}^1(\mathbb{F}_q)$ curve
- start at v_0 with an algebro-geometric code on $\mathbb{P}^1(\mathbb{F}_q)$
- propagate along the tree with other Reed-Solomon codes at vertices taking some of input from previous vertices
- build code with inputs at vertices of $\mathcal{T}_{\mathbb{K}}$ and outputs at boundary $\mathbb{P}^1(\mathbb{K})$
- pass from classical to quantum codes using Calderbank–Rains–Shor–Sloane algorithm

Algebro-geometric codes

- algebraic points $X(\mathbb{F}_q)$ of a curve X over a finite field $\mathbb{F}_q$
- set $A \subset X(\mathbb{F}_q)$ and divisor D on X with $supp(D) \cap A = \emptyset$
- code $C = C_X(A, D)$ by evaluation at A of rational functions $f \in \mathbb{F}_q(X)$ with poles at D
- bound on order of pole of f at D determines dimension of the linear code

Reed-Solomon codes case $X(\mathbb{F}_q) = \mathbb{P}^1(\mathbb{F}_q)$

- $C = \{(f(x_1), \cdots, f(x_n)) \,:\, f \in \mathbb{F}_q[x], \deg(f) < k\}$ gives an $[n, k, n-k+1]_q$ with $n \leq q$
- or homogeneous polynomials at points $x_i = (u_i : v_i) \in \mathbb{P}^1(\mathbb{F}_q)$

$$\hat{C} = \{(f(u_1, v_1), \ldots, f(u_n, v_n)) \,:\, f \in \mathbb{F}_q[u, v], \text{homog. } \deg(f) < k\}$$

- generalized Reed-Solomon codes: $w = (w_1, \ldots, w_n) \in \mathbb{F}_q^n$

$$C_{w,k} = \{(w_1 f(x_1), \cdots, w_n f(x_n)) \,:\, f \in \mathbb{F}_q[x], \deg(f) < k\}$$

$$\hat{C}_{w,k} = \{(w_1 f(u_1, v_1), \ldots, w_n f(u_n, v_n)) \,:\, f \in \mathbb{F}_q[u, v], \text{homog.} \deg(f) < k\}$$

Reed-Solomon codes on $\mathcal{T}_{\mathbb{K}}$

- encode inputs at each vertex to output at boundary $\mathbb{P}^1(\mathbb{K})$
- algebraic points $\mathbb{P}^1(\mathbb{F}_q)$ of reduction curve (at $v_0 \in \mathcal{T}_{\mathbb{K}}$)
- Reed-Solomon code $\hat{C}_{w,k}$ maximal length $n = q + 1$
- k-tuple of q-ary bits $a = (a_0, \ldots, a_{k-1}) \in \mathbb{F}_q^k$, output q-ary bit $f_a(u_j, v_j) \in \mathbb{F}_q$ at each point $x_j = (u_j : v_j) \in \mathbb{P}^1(\mathbb{F}_q)$ with $f_a(u, v) = \sum_{i=0}^{k} a_i u^i v^{k-1-i}$
- inductively at next vertices outward choice of root vertex v_0 specifies one preferred edge at v (point $\infty \in \mathbb{P}^1(\mathbb{F}_q)$); next input $a = (a_1, \ldots, a_{k-1}) \in \mathbb{F}_q^{k-1}$ with a_0 the q-bit deposited at ∞ point by previous code

From classical to quantum codes   CRSS algorithm

- $\mathcal{H} = \mathbb{C}^q$ single q-ary qubit, o.n. basis $|a\rangle$ with $a \in \mathbb{F}_q$
- Quantum error correcting codes: subspaces $\mathcal{C} \subset \mathcal{H}_n = \mathcal{H}^{\otimes n}$ error correcting for up to d "q-ary bit flip" and "phase flip" errors $E = E_1 \otimes \cdots \otimes E_n$, $\omega(E) = \#\{i \,:\, E_i \neq I\} < d$

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E \, P_{\mathcal{C}}$$

  orthogonal projection $P_{\mathcal{C}}$ onto $\mathcal{C}$
- q-ary bit flip and phase flip on $\mathbb{C}^p$: $TR = \xi RT$ with $\xi^p = 1$

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & & & & \\ & \xi & & & \\ & & \xi^2 & & \\ & & & \ddots & \\ & & & & \xi^{p-1} \end{pmatrix}$$

Quantum Error Operators

- $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ trace function $\mathrm{Tr}(a) = \sum_{i=0}^{r-1} a^{p^i}$
$$T_b|a\rangle = |a+b\rangle, \quad R_b|a\rangle = \xi^{\mathrm{Tr}(ab)}|a\rangle,$$
- $b \in \mathbb{F}_q$ as an $\mathbb{F}_p$-vector space, $q = p^r$
$$T_a := T^{a_1} \otimes \cdots \otimes T^{a_r}, \quad R_b := R^{b_1} \otimes \cdots \otimes R^{b_r}$$
- $T_a R_b$, $a, b \in \mathbb{F}_q$, o.n. basis $M_{q\times q}(\mathbb{C})$ for $\langle A, B\rangle = \mathrm{Tr}(A^* B)$, generate all possible quantum errors on $\mathcal{H} = \mathbb{C}^q$
- error operators $E_{a,b}$ with $E^p_{a,b} = I$
$$E_{a,b} = T_a R_b = (T_{a_1} \otimes \cdots \otimes T_{a_n})(R_{b_1} \otimes \cdots \otimes R_{b_n})$$
for $a = (a_1, \ldots, a_n), b = (b_1 \ldots, b_n) \in \mathbb{F}_q^n$
- commutation and composition rules
$$E_{a,b} E_{a',b'} = \xi^{\langle a,b'\rangle - \langle b,a'\rangle} E_{a',b'} E_{a,b}$$
$$E_{a,b} E_{a',b'} = \xi^{-\langle b,a'\rangle} E_{a+a',b+b'},$$
where $\langle a, b\rangle = \sum_i \langle a_i, b_i\rangle = \sum_{i,j} a_{i,j} b_{i,j}$, with $a_i = (a_{i,j}), b_i = (b_{i,j}) \in \mathbb{F}_q$ identified with $\mathbb{F}_p$-vector space

Quantum stabilizer codes

- group $\mathcal{G}_n = \{\xi^i E_{a,b},\ a, b \in \mathbb{F}_q^n,\ 0 \le i \le p-1\}$ order pq^{2n}
- quantum stabilizer error-correcting code $\mathcal{C} \subset \mathcal{H}_n$ joint eigenspace of operators $E_{a,b}$ in an abelian subgroup $\mathcal{S} \subset \mathcal{G}_n$
- $\varphi \in \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_p^r)$ automorphism

$$\langle (a,b),(a',b') \rangle = \langle a, \varphi(b') \rangle - \langle a', \varphi(b) \rangle$$

- $C \subset \mathbb{F}_q^{2n}$ is a classical self-orhogonal code with respect to this pairing $\Rightarrow$ subgroup $\mathcal{S} \subset \mathcal{G}_n$ of $\xi^i E_{a,\varphi(b)}$ with $(a,b) \in C$ is abelian
- CRSS algorithm associates to self-orthogonal classical $[2n,k,d]_q$ code C stabilizer quantum $[[n, n-k, d_Q]]_q$-code

$$d_Q = \min\{\omega(a,b)\ :\ (a,b) \in C^\perp \smallsetminus C\}$$

$\omega(a,b) = \#\{i\ :\ a_i \neq 0 \text{ or } b_i \neq 0\}$ and
$C^\perp = \{(v,w) \in \mathbb{F}_q^{2n}\ :\ \langle (a,b),(v,w) \rangle = 0,\ \forall (a,b) \in C\}$

Holographic quantum codes on $\mathcal{T}_{\mathbb{K}}$

- Hermitian self-dual case: $\langle v, w \rangle_H = \sum_{i=1}^n v_i w_i^q$, with $v, w \in \mathbb{F}_{q^2}^n$
- Hermitian-self-dual length n over $\mathbb{F}_{q^2}$ gives self-dual code $\tilde{C}$ length $2n$ over $\mathbb{F}_q$ then CRSS
- Hermitian self-duality conditions for generalized Reed–Solomon codes with $w = (w_1, \ldots, w_n) \in (\mathbb{F}_{q^2}^*)^n$
- For $w_i = 1$ and $n = q^2$ with $k = q$, Hermitian-self-dual Reed-Solomon code $C = C_{1,q}$ and associated $[[q^2 + 1, q^2 - 2q + 1, q + 1]]_q$-quantum Reed-Solomon code $\mathcal{C}$
- Extension $\mathbb{L}$ with residue field $\mathbb{F}_{q^2}$, at each vertex of $\mathcal{T}_{\mathbb{L}}$ quantum Reed-Solomon $[[q^2 + 1, q^2 - 2q + 1, q + 1]]_q$-code $\mathcal{C}$
- a q-ary qubit stored at each of the legs surrounding vertex in $\mathcal{T}_{\mathbb{L}}$
- quantum code $\mathcal{C}$ corrects quantum errors of weight up to $q + 1$: set of directions along the subtree $\mathcal{T}_{\mathbb{K}}$

Further work (in progress)

- lifting bulk geometry (gravity and holographic codes) from Bruhat–Tits tree to Drinfeld p-adic upper half plane
- higher dimensional holography on Bruhat–Tits buildings of $\mathrm{GL}_n(\mathbb{Q}_p)$ and p-adic symmetric spaces
- perturbative quantum field theory (Feynman diagrams) in p-adic setting, theories with holographic dual