Feynman integrals, singular hypersurfaces, and motives

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Quantum Field Theory perturbative (massless) scalar field theory

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density (Euclidean)

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \hat{\phi}(p_1) \cdots \hat{\phi}(p_N) U(\Gamma(p_1, \ldots, p_N)) d^D p_1 \cdots d^D p_N$$

$$U(\Gamma(p_1, \ldots, p_N)) = \int I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N) d^D k_1 \cdots d^D k_\ell$$

$\ell = b_1(\Gamma)$ loops

Feynman rules for $I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N)$:

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in $\mathcal{L}$)

$$\sum_{e_i \in E(\Gamma): s(e_i) = v} k_i = 0$$

- Integration over k_i, internal edges

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

$n = \#E_{int}(\Gamma)$, $N = \#E_{ext}(\Gamma)$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

Parametric Feynman integrals

- Schwinger parameters $q_1^{-k_1} \cdots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1 - 1} \cdots s_n^{k_n - 1} \, ds_1 \cdots ds_n.$$

- Feynman trick

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n - D\ell/2}}$$

$\sigma_n = \{ t \in \mathbb{R}_+^n \,|\, \sum_i t_i = 1 \}$, vol form ω_n

Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e \quad \text{with} \quad (M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik} \eta_{ir}$$

Massless case $m = 0$:

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (complement of spanning tree plus one edge)
$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$ for $\sum_{e \in E_{ext}(\Gamma)} p_e = 0$
with $\deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

stable range $-n + D\ell/2 \geq 0$; log divergent $n = D\ell/2$:

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}$$

Graph hypersurfaces

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

Graph hypersurfaces $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \,|\, \Psi_\Gamma(t) = 0\} \quad \deg = b_1(\Gamma)$$

• Relative cohomology: (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)) \quad \text{with} \quad \Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$$

• Periods: $\int_\sigma \omega$ integrals of algebraic differential forms ω on a cycle σ defined by algebraic equations in an algebraic variety

Feynman integrals and periods

Parametric Feynman integral: algebraic differential form on cycle in algebraic variety

But... divergent: where $X_\Gamma \cap \sigma_n \neq \emptyset$, inside divisor $\Sigma_n \supset \sigma_n$ of coordinate hyperplanes

- Blowups of coordinate linear spaces defined by edges of 1PI subgraphs (toric variety $P(\Gamma)$)
- Iterated blowup $P(\Gamma)$ separates strict transform of X_Γ from non-negative real points
- Deform integration chain: monodromy problem; lift to $P(\Gamma)$
- Subtraction of divergences: Poincaré residuces and limiting mixed Hodge structure

• S. Bloch, E. Esnault, D. Kreimer, *On motives associated to graph polynomials*, arXiv:math/0510011.

• S. Bloch, D. Kreimer, *Mixed Hodge Structures and Renormalization in Physics*, arXiv:0804.4399.

Motives of algebraic varieties (Grothendieck) Universal cohomology theory for algebraic varieties (with realizations)

Mixed motives: varieties that are possibly singular or not projective (much more complicated theory than pure (smooth projective)!)
Triangulated category $\mathcal{DM}$ (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

Mixed Tate motives: $\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$
Tate object: $\mathbb{Q}(1)$ formal inverse of Lefschetz motive $\mathbb{L} = h^2(\mathbb{P}^1)$

Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Periods and motives: Constraints on numbers obtained as periods from the motive of the variety!

• Periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[1/(2\pi i)]$-combinations of Multiple Zeta Values

$$\zeta(k_1, k_2, \dots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r}$$

Conjecture proved recently:

• Francis Brown, *Mixed Tate motives over* $\mathbb{Z}$, arXiv:1102.1312.

Feynman integrals and periods: MZVs as *typical* outcome:

• D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, arXiv:hep-th/9609128

⇒ Conjecture (Kontsevich 1997): Motives of graph hypersurfaces are mixed Tate (or counting points over finite fields behavior)

Conjecture was first verified for all graphs up to 12 edges:

• J. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, 1998

But ... Conjecture is false!

- P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, arXiv:math/0012198
- Dzmitry Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
- Francis Brown, Oliver Schnetz, *A K3 in phi4*, arXiv:1006.4064.
- Francis Brown, Dzmitry Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056

• Belkale–Brosnan: general argument shows "motives of graph hypersurfaces can be arbitrarily complicated"

• Doryn, Brown–Schnetz, Brown–Doryn: explicit counterexamples (14 edges)

Motives and the Grothendieck ring of varieties

• Difficult to determine explicitly the motive of X_Γ (singular variety!) in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X_\Gamma]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \setminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$
(K_0 group of category of pure motives: virtual motives)

Universal Euler characteristics:

Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathcal{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathcal{V}) \to \mathcal{R}$$

Examples:

- Topological Euler characteristic
- Couting points over finite fields
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Universality: a dichotomy

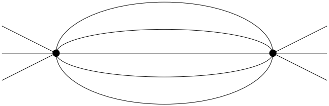
- After localization (Belkale-Brosnan): the graph hypersurfaces X_Γ *generate* the Grothendieck ring *localized* at $\mathbb{L}^n - \mathbb{L}$, $n > 1$
- Stable birational equivalence: the graph hypersurfaces span $\mathbb{Z}$ inside $\mathbb{Z}[SB] = K_0(\mathcal{V})|_{\mathbb{L}=0}$

• P. Aluffi, M.M. *Graph hypersurfaces and a dichotomy in the Grothendieck ring*, arXiv:1005.4470

Graph hypersurfaces: computing in the Grothendieck ring

• P. Aluffi, M.M. *Feynman motives of banana graphs*, arXiv:0807.1690

Example: *banana graphs* $\Psi_\Gamma(t) = t_1 \cdots t_n(\frac{1}{t_1} + \cdots + \frac{1}{t_n})$



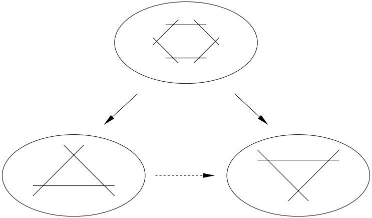
$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive and $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$
$X_{\Gamma^\vee} = \mathcal{L}$ hyperplane in $\mathbb{P}^{n-1}$
$\Gamma^\vee =$ dual graph $=$ polygon

Method: Dual graph and Cremona transformation

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto (\frac{1}{t_1} : \cdots : \frac{1}{t_n})$$

outside $\mathcal{S}_n$ singularities locus of $\Sigma_n = \{\prod_i t_i = 0\}$, ideal $I_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \cdots, t_1 t_3 \cdots t_n)$



$$\Psi_\Gamma(t_1, \ldots, t_n) = (\prod_e t_e) \Psi_{\Gamma^\vee}(t_1^{-1}, \ldots, t_n^{-1})$$

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)$$

isomorphism of X_Γ and $X_{\Gamma^\vee}$ outside of Σ_n

For banana graph case obtain:

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}$$

$$X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n \quad \text{with} \quad [\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}$$

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n]$$

Using Cremona transformation: $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$

In particular get topological information on the X_{Γ_n}
$\Rightarrow \chi(X_{\Gamma_n}) = n + (-1)^n$

Sum over graphs

Even when non-planar: can transform by Cremona (new hypersurface, not of dual graph)

$\Rightarrow$ graphs by removing edges from complete graph: fixed vertices

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)} \in \mathbb{Z}[\mathbb{L}],$$

Tate motive (though $[X_\Gamma]$ individually need not be)

- Spencer Bloch, *Motives associated to sums of graphs*, arXiv:0810.1313

Suggests that although individual graphs need not give mixed Tate contribution, the sum over graphs in Feynman amplitudes (fixed loops, not vertices) may be mixed Tate

Deletion–contraction relation

In general cannot compute explicitly $[X_\Gamma]$: would like relations that simplify the graph... but cannot have *true* deletion-contraction relation, else always mixed Tate... What kind of deletion-contraction?

- P. Aluffi, M.M. *Feynman motives and deletion-contraction relations*, arXiv:0907.3225

- Graph polynomials: Γ with $n \geq 2$ edges, $\deg \Psi_\Gamma = \ell > 0$

$$\Psi_\Gamma = t_e \Psi_{\Gamma \smallsetminus e} + \Psi_{\Gamma / e}$$

$$\Psi_{\Gamma \smallsetminus e} = \frac{\partial \Psi_\Gamma}{\partial t_n} \quad \text{and} \quad \Psi_{\Gamma/e} = \Psi_\Gamma|_{t_n = 0}$$

- General fact: $X = \{\psi = 0\} \subset \mathbb{P}^{n-1}$, $Y = \{F = 0\} \subset \mathbb{P}^{n-2}$

$$\psi(t_1, \dots, t_n) = t_n F(t_1, \dots, t_{n-1}) + G(t_1, \dots, t_{n-1})$$

$\overline{Y}$ = cone of Y in $\mathbb{P}^{n-1}$: Projection from $(0 : \cdots : 0 : 1) \Rightarrow$ isomorphism

$$X \smallsetminus (X \cap \overline{Y}) \xrightarrow{\sim} \mathbb{P}^{n-2} \smallsetminus Y$$

Then deletion-contraction: for $\hat{X}_\Gamma \subset \mathbb{A}^n$

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus (\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}]$$

if e not a bridge or a looping edge

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma/e}]$$

if e bridge

$$\begin{aligned} [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}] \\ &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma/e}] \end{aligned}$$

if e looping edge

Note: intersection $\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e}$ difficult to control motivically: first place where non-Tate contributions will appear

Example of application: Multiplying edges

Γ_{me} obtained from Γ by replacing edge e by m parallel edges
($\Gamma_{0e} = \Gamma \smallsetminus e$, $\Gamma_e = \Gamma$)

Generating function: $\mathbb{T} = [\mathbb{G}_m] \in K_0(\mathcal{V})$

$$\sum_{m\geq 0} \mathbb{U}(\Gamma_{me}) \frac{s^m}{m!} = \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma) + \frac{e^{\mathbb{T}s} + \mathbb{T}e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma \smallsetminus e) + \left(s\, e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \right) \mathbb{U}(\Gamma / e).$$

e not bridge nor looping edge: similar for other cases

For doubling: inclusion-exclusion

$$\mathbb{U}(\Gamma_{2e}) = \mathbb{L} \cdot [\mathbb{A}^n \smallsetminus (\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o})] - \mathbb{U}(\Gamma)$$
$$[\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o}] = [\hat{X}_{\Gamma/e}] + (\mathbb{L}-1) \cdot [\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e}]$$

then cancellation

$$\mathbb{U}(\Gamma_{2e}) = (\mathbb{L}-2) \cdot \mathbb{U}(\Gamma) + (\mathbb{L}-1) \cdot \mathbb{U}(\Gamma \smallsetminus e) + \mathbb{L} \cdot \mathbb{U}(\Gamma/e)$$

Example of application: Lemon graphs and chains of polygons

$\Lambda_m =$ lemon graph m wedges; $\Gamma^{\Lambda}_m =$ replacing edge e of Γ with Λ_m

Generating function: $\sum_{m\geq 0} \mathbb{U}(\Gamma^{\Lambda}_m) s^m =$

$$\frac{(1-(\mathbb{T}+1)s)\,\mathbb{U}(\Gamma) + (\mathbb{T}+1)\mathbb{T}s\,\mathbb{U}(\Gamma \smallsetminus e) + (\mathbb{T}+1)^2 s\,\mathbb{U}(\Gamma/e)}{1-\mathbb{T}(\mathbb{T}+1)s - \mathbb{T}(\mathbb{T}+1)^2 s^2}$$

e not bridge or looping edge; similar otherwise

Recursive relation:

$$\mathbb{U}(\Lambda_{m+1}) = \mathbb{T}(\mathbb{T}+1)\mathbb{U}(\Lambda_m) + \mathbb{T}(\mathbb{T}+1)^2\mathbb{U}(\Lambda_{m-1})$$

$a_m = \mathbb{U}(\Lambda_m)$ is a *divisibility sequence*: $\mathbb{U}(\Lambda_{m-1})$ divides $\mathbb{U}(\Lambda_{n-1})$ if m divides n

Determinant hypersurfaces and Schubert cells

Mixed Tate question reformulated in terms of determinant hypersurfaces and intersections of unions of Schubert cells in flag varieties

• P. Aluffi, M.M. *Parametric Feynman integrals and determinant hypersurfaces*, arXiv:0901.2107

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik}\eta_{ir}, \quad \hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1) \Rightarrow \text{ mixed Tate}$$

When Υ embedding

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}$$

If $\hat{\Sigma}_\Gamma$ normal crossings divisor in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \quad \text{mixed Tate motive?}$$

Combinatorial conditions for embedding $\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$

- Closed 2-cell embedded graph $\iota : \Gamma \hookrightarrow S_g$ with $S_g \smallsetminus \Gamma$ union of open disks (faces); closure of each is a disk.
- Two faces have at most one edge in common
- Every edge in the boundary of two faces

Sufficient: Γ 3-edge-connected with closed 2-cell embedding of face width ≥ 3.

Face width: largest $k \in \mathbb{N}$, every non-contractible simple closed curve in S_g intersects Γ at least k times (∞ for planar).

Note: 2-edge-connected =1PI; 2-vertex-connected conjecturally implies face width ≥ 2

Identifying the motive $\mathfrak{m}(X, Y)$. Set $\hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{\ell,g}$ $(f = \ell - 2g + 1)$

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

$$\begin{cases} x_{ij} & = 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} & = 0 \quad 1 \leq i \leq f-1 \end{cases}$$

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell))$$

$\hat{\Sigma}_{\ell,g}$ = normal crossings divisor $\Upsilon_\Gamma(\partial\sigma_n) \subset \hat{\Sigma}_{\ell,g}$
depends only on $\ell = b_1(\Gamma)$ and g = min genus of S_g

- Sufficient condition: Varieties of frames mixed Tate?

$$\mathbb{F}(V_1, \ldots, V_\ell) := \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \mid v_k \in V_k\}$$

Varieties of frames

- Two subspaces: $(d_{12} = \dim(V_1 \cap V_2))$

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$$

- Three subspaces $(D = \dim(V_1 + V_2 + V_3))$

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$

$$-(\mathbb{L}-1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1)$$

$$+(\mathbb{L}-1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L}-1)^3$$

- Higher: difficult to find suitable induction
- Other formulation: $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ locus of complete flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$, with $\dim E_i \cap V_i = d_i$ and $\dim E_i \cap V_{i+1} = e_i$: are these mixed Tate? (for all choices of d_i, e_i)
- $\mathbb{F}(V_1, \ldots, V_\ell)$ fibration over $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$: class $[\mathbb{F}(V_1, \ldots, V_\ell)]$

$$= [Flag_{\ell,\{d_i,e_i\}}(\{V_i\})](\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - \mathbb{L}^{e_1})(\mathbb{L}^{d_3} - \mathbb{L}^{e_2}) \cdots (\mathbb{L}^{d_r} - \mathbb{L}^{e_{r-1}})$$

$Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ intersection of unions of Schubert cells in flag varieties
$\Rightarrow$ Kazhdan–Lusztig?

Different approach to regularization and renormalization

• Based on ongoing work with Xiang Ni

Main ingredients:

- Algebraic renormalization (Hopf algebras and Rota–Baxter algebras)
- Hypersurfaces and Rota–Baxter algebras of meromorphic forms
- Forms with logarithmic poles and Leray residues
- Wonderful compactifications

Developed for Feynman integrals in configuration spaces in
• O. Ceyhan, M.M. *Algebraic renormalization and Feynman integrals in configuration spaces*, arXiv:1308.5687

Regularization and renormalization

Removing divergences from Feynman integrals by adjusting bare parameters in the Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2(1-\delta Z) + \left(\frac{m^2-\delta m^2}{2}\right)\phi^2 - \frac{g+\delta g}{6}\phi^3$$

Regularization: replace divergent integral $U(\Gamma)$ by function $U^z(\Gamma)$ with pole ($z \in \mathbb{C}^*$ in DimReg, ϵ deformation of X_Γ, etc.)
Renormalization: consistency over subgraphs (Hopf algebra structure)

- Kreimer, Connes–Kreimer, Connes–M.: Hopf algebra of Feynman graphs and BPHZ renormalization method in terms of Birkhoff factorization and differential Galois theory

- Ebrahimi-Fard, Guo, Kreimer: algebraic renormalization in terms of Rota–Baxter algebras

BPHZ renormalization method:

- Preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

- Counterterm: projection onto polar part

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$

- Renormalized value:

$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

Connes–Kreimer Hopf algebra $\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depends on theory $\mathcal{L}(\phi)$)

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom): Ward identities as Hopf ideals

Algebraic renormalization (Ebrahimi-Fard, Guo, Kreimer)

• Rota–Baxter algebra of weight $\lambda = -1$: $\mathcal{R}$ commutative unital algebra; $T : \mathcal{R} \to \mathcal{R}$ linear operator with

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy)$$

• Example: $T =$ projection onto polar part of Laurent series

• T determines splitting $\mathcal{R}_+ = (1 - T)\mathcal{R}$, $\mathcal{R}_- =$ unitization of $T\mathcal{R}$; both $\mathcal{R}_\pm$ are algebras

• Feynman rule $\phi : \mathcal{H} \to \mathcal{R}$ commutative algebra homomorphism from CK Hopf algebra $\mathcal{H}$ to Rota–Baxter algebra $\mathcal{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$$

• Note: ϕ does *not know* that $\mathcal{H}$ Hopf and $\mathcal{R}$ Rota-Baxter, only commutative algebras

- Birkhoff factorization $\exists \phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm)$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$

- Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X''))$$

$$\phi_+(X) = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$

Example of algebraic renormalization (Connes–Kreimer):

- Dimensional Regularization: $U_\mu^z(\Gamma(p_1,\ldots,p_N))$

$$= \int \mu^{z\ell} d^{D-z}k_1 \cdots d^{D-z}k_\ell I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N)$$

Laurent series in $z \in \Delta^* \subset \mathbb{C}^*$

- Rota–Baxter algebra: T = projection onto polar part of Laurent series
- loop $= \phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$ (germs of meromorphic functions)
- Feynman integral $U(\Gamma) = \phi(\Gamma)$

counterterms $C(\Gamma) = \phi_-(\Gamma)$

renormalized value $R(\Gamma) = \phi_+(\Gamma)|_{z=0}$

Rota–Baxter algebras of meromorphic forms

smooth hypersurface $Y = \{f = 0\}$ in $\mathbb{P}^n$

- $\mathcal{M}^\star_{\mathbb{P}^n,Y}$ = meromorphic forms, poles (arbitrary order) on Y

$$\omega = \sum_{p \geq 0} \frac{\alpha_p}{f^p} \mapsto T(\omega) = \sum_{p \geq 1} \frac{\alpha_p}{f^p}$$

Rota–Baxter (graded) algebra of weight -1

$$T(x)T(y) = T(xT(y)) + T(T(x)y) - T(xy)$$

- Restrict to $\Omega^\star_{\mathbb{P}^n}(\log(Y))$ forms with log poles:

$$\omega = \frac{df}{f} \wedge \xi + \eta \mapsto T(\omega) = \frac{df}{f} \wedge \xi$$

Rota–Baxter identity becomes

$$T(xy) = T(xT(y)) + T(T(x)y) = xT(y) + T(x)y$$

hence T is a *derivation*

Pole subtraction: $\omega \mapsto (1-T)\omega$

Vanishing Leray residue $\omega = d\log(f) \wedge \xi + \eta$

$$\mathrm{Res}_Y(\omega) = \xi$$

holomorphic form on X

Can extend to:

- Smooth hypersurface Y in a smooth projective X;
- Normal crossings divisor Y in a smooth projective X;
- Singular hypersurface Y in a smooth projective X: using Saito's forms with log poles and residues

$$h\omega = \frac{df}{f} \wedge \xi + \eta, \quad \mathrm{Res}_Y(\omega) = \frac{1}{h}\xi$$

General strategy for Feynman integrals

• (graded) Hopf algebra of Feynman graphs
$\Gamma_1 \cdot \Gamma_2 = (-1)^{\# E(\Gamma_1) \# E(\Gamma_2)} \Gamma_2 \cdot \Gamma_1$

• Fixed number of loops ℓ: a smooth projective variety X_ℓ and a (singular) hypersurface $Y_\ell \subset X_\ell$, such that the motive $m(X_\ell)$ is mixed Tate

• A morphism of graded algebras $\phi : \mathcal{H} \to \mathcal{M}^*_{X_\ell, Y_\ell}$

$$\phi(\Gamma) = \eta_\Gamma$$

algebraic differential form on X_ℓ with polar locus Y_ℓ

• Rota–Baxter operator T (polar part) on $\mathcal{M}^*_{X_\ell, Y_\ell}$

$\Rightarrow$ Birkhoff decomposition $\phi_\pm$ gives holomorphic form $\phi_+(\Gamma)$ on X_ℓ

$$\int_\sigma \phi_+(\Gamma)$$

is a *period* of a mixed Tate motive (always)

Especially nice situation:

When all cohomology classes of $H^*(X_\ell \smallsetminus Y_\ell)$ can be represented by forms with logarithmic poles

Examples:

- Normal crossings divisors (Deligne)
- Locally quasi-homogeneous free divisors (F. J. Castro-Jiménez, D. Mond, and L. Narvaéz-Macarro)

Then can use restriction of Rota–Baxter operator T to forms with log poles $\Omega^*_{X_\ell}(\log(Y_\ell))$

$\Rightarrow$ The Birkhoff factorization formula simplifies drastically
(no correction terms from subdivergences, only pole subtraction)

Application to parametric Feynman integrals

Assume $n \geq (\ell+1)D/2$ and consider algebraic differential form (take $p \in \mathbb{Q}$)

$$\eta_\Gamma = \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n(\ell+1)D/2}}$$

on $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell = \mathrm{GL}_\ell$

$$\phi(\Gamma) = \eta_\Gamma \in \mathcal{M}^*_{\mathbb{P}^{\ell^2-1},\mathcal{D}_\ell}$$

apply Birkhoff factorization and evaluate convergent integral

$$\int_{\Sigma_{g,\ell}} \phi_+(\Gamma)$$

of algebraic form $\phi_+(\Gamma)$.

Kausz compactification

better method: reduce to forms with logarithmic poles

Need a better *compactification* of GL_ℓ

• PGL_ℓ has a wonderful compactification $\overline{\mathrm{PGL}_\ell}$ in the sense of DeConcini–Procesi (Vainsencher)

• Iterated blowup description: $X_0 = \mathbb{P}^{\ell^2-1}$, loci Y_i matrices rank i, with $\bar{Y}_i$ closure in X_{i-1}

$$X_i = \mathrm{Bl}_{\bar{Y}_i}(X_{i-1})$$

$X_{\ell-1} = \overline{\mathrm{PGL}_\ell}$ smooth;
Y_i are PGL_i-bundles over a product of Grassmannians

Kausz compactification $K\mathrm{GL}_\ell$:

• Kausz compactification = closure of GL_ℓ inside wonderful compactification of $\mathrm{PGL}_{\ell+1}$

• Iterated blowup with $\mathcal{X}_0 = \mathbb{P}^{\ell^2}$,

$$\mathcal{X}_i = \mathrm{Bl}_{\mathcal{Y}_{i-1}\cup\mathcal{H}_i}(\mathcal{X}_{i-1})$$

with $\mathcal{Y}_i \subset \mathbb{A}^{\ell^2}$ matrices rank i and $\mathcal{H}_i$ matrices at infinity, in $\mathbb{P}^{\ell^2-1} = \mathbb{P}^{\ell^2} \smallsetminus \mathbb{A}^{\ell^2}$

• the $\mathcal{X}_i$ are smooth and blowup loci disjoint unions of $\overline{\mathrm{PGL}_i}$-bundles and$K\mathrm{GL}_i$-bundles over a product of Grassmannians

• complement of GL_ℓ in $K\mathrm{GL}_\ell$ is normal crossings divisor

I. Kausz, *A modular compactification of the general linear group*, Documenta Math. 5 (2000) 553–594

Motive of the Kausz compactification $m(K\mathrm{GL}_\ell)$

- Chow motive of a blowup along a smooth locus (Manin)

$$m(Bl_Y(X)) = m(X) \oplus \bigoplus_{r=1}^{codim(Y)-1} m(Y) \otimes \mathbb{L}^{\otimes r},$$

- motives of Grassmannians $G(d,n)$ (Köck)

$$m(G(d,n)) = \bigoplus_{\lambda \in W^d} \mathbb{L}^{\otimes |\lambda|}$$

$$W^d = \{\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{N}^d \mid n - d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0\}$$

and $|\lambda| = \sum_i \lambda_i$

then inductively:

• motive of a $\overline{\mathrm{PGL}_i}$-bundle over a product of Grassmannians: has a "sufficiently good" cell decomposition so that motive of F bundle B over Z decomposes as a product

$$m(B) \simeq m(F) \otimes m(Z)$$

• for $K\mathrm{GL}_i$-bundles over products of Grassmannians also show inductively that have good cell decomposition

Conclusion 1: the motive $m(K\mathrm{GL}_\ell)$ is mixed Tate

Conclusion 2: the renormalized Feynman integral

$$\int_{\pi^{-1}(\Sigma_{g,\ell})} (1-T)\eta_\Gamma$$

is a period of $K\mathrm{GL}_\ell$

... but information loss for certain graphs